LY

IN
NN
E

C. V.
pub-
Uni-

LPH,

ell as
o the
N. Y.

RFER,
ENTS,
L. R.
NSON,
M. S.
LLEY,
DEN,

Y.

A HALF-CENTURY OF MATHEMATICS

HERMANN WEYL, Institute for Advanced Study

1. Introduction. Axiomatics. Mathematics, beside astronomy, is the oldest of all sciences. Without the concepts, methods and results found and developed by previous generations right down to Greek antiquity, one cannot understand either the aims or the achievements of mathematics in the last fifty years. Mathematics has been called the science of the infinite; indeed, the mathematician invents finite constructions by which questions are decided that by their very nature refer to the infinite. That is his glory. Kierkegaard once said religion deals with what concerns man unconditionally. In contrast (but with equal exaggeration) one may say that mathematics talks about the things which are of no concern at all to man. Mathematics has the inhuman quality of starlight, brilliant and sharp, but cold. But it seems an irony of creation that man's mind knows how to handle things the better the farther removed they are from the center of his existence. Thus we are cleverest where knowledge matters least: in mathematics, especially in number theory. There is nothing in any other science that, in subtlety and complexity, could compare even remotely with such mathematical theories as for instance that of algebraic class fields. Whereas physics in its development since the turn of the century resembles a mighty stream rushing on in one direction, mathematics is more like the Nile delta, its waters fanning out in all directions. In view of all this: dependence on a long past, other-worldliness, intricacy, and diversity, it seems an almost hopeless task to give a non-esoteric account of what mathematicians have done during the last fifty years. What I shall try to do here is, first to describe in somewhat vague terms general trends of development, and then in more precise language explain the most outstanding mathematical notions devised, and list some of the more important problems solved, in this period.

One very conspicuous aspect of twentieth century mathematics is the enormously increased role which the axiomatic approach plays. Whereas the axiomatic method was formerly used merely for the purpose of elucidating the foundations on which we build, it has now become a tool for concrete mathematical research. It is perhaps in algebra that it has scored its greatest successes. Take for instance the system of real numbers. It is like a Janus head facing in two directions: on the one side it is the field of the algebraic operations of addition and multiplication; on the other hand it is a continuous manifold, the parts of which are so connected as to defy exact isolation from each other. The one is the algebraic, the other the topological face of numbers. Modern axiomatics, simple-minded as it is (in contrast to modern politics), does not like such ambiguous mixtures of peace and war, and therefore cleanly separated both aspects from each other.

In order to understand a complex mathematical situation it is often convenient to separate in a natural manner the various sides of the subject in question, make each side accessible by a relatively narrow and easily surveyable

group of notions and of facts formulated in terms of these notions, and finally return to the whole by uniting the partial results in their proper specialization. The last synthetic act is purely mechanical. The art lies in the first, the analytic act of suitable separation and generalization. Our mathematics of the last decades has wallowed in generalizations and formalizations. But one misunderstands this tendency if one thinks that generality was sought merely for generality's sake. The real aim is simplicity: every natural generalization simplifies since it reduces the assumptions that have to be taken into account. It is not easy to say what constitutes a natural separation and generalization. For this there is ultimately no other criterion but fruitfulness: the success decides. In following this procedure the individual investigator is guided by more or less obvious analogies and by an instinctive discernment of the essential acquired through accumulated previous research experience. When systematized the procedure leads straight to axiomatics. Then the basic notions and facts of which we spoke are changed into undefined terms and into axioms involving them. The body of statements deduced from these hypothetical axioms is at our disposal now, not only for the instance from which the notions and axioms were abstracted, but wherever we come across an interpretation of the basic terms which turns the axioms into true statements. It is a common occurrence that there are several such interpretations with widely different subject matter.

The axiomatic approach has often revealed inner relations between, and has made for unification of methods within, domains that apparently lie far apart. This tendency of several branches of mathematics to coalesce is another conspicuous feature in the modern development of our science, and one that goes side by side with the apparently opposite tendency of axiomatization. It is as if you took a man out of a milieu in which he had lived not because it fitted him but from ingrained habits and prejudices, and then allowed him, after thus setting him free, to form associations in better accordance with his true inner nature.

In stressing the importance of the axiomatic method I do not wish to exaggerate. Without inventing new constructive processes no mathematician will get very far. It is perhaps proper to say that the strength of modern mathematics lies in the interaction between axiomatics and construction. Take algebra as a representative example. It is only in this century that algebra has come into its own by breaking away from the one universal system Ω of numbers which used to form the basis of all mathematical operations as well as all physical measurements. In its newly-acquired freedom algebra envisages an infinite variety of "number fields" each of which may serve as an operational basis; no attempt is made to embed them into the one system Ω. Axioms limit the possibilities for the number concept; constructive processes yield number fields that satisfy the axioms.

In this way algebra has made itself independent of its former master analysis and in some branches has even assumed the dominant role. This development in mathematics is paralleled in physics to a certain degree by the transition from

classical to quantum physics, inasmuch as the latter ascribes to each physical structure its own system of observables or quantities. These quantities are subject to the algebraic operations of addition and multiplication; but as their multiplication is non-commutative, they are certainly not reducible to ordinary numbers.

At the International Mathematical Congress in Paris in 1900 David Hilbert, convinced that problems are the life-blood of science, formulated twenty-three unsolved problems which he expected to play an important role in the development of mathematics during the next era. How much better he predicted the future of mathematics than any politician foresaw the gifts of war and terror that the new century was about to lavish upon mankind! We mathematicians have often measured our progress by checking which of Hilbert's questions had been settled in the meantime. It would be tempting to use his list as a guide for a survey like the one attempted here. I have not done so because it would necessitate explanation of too many details. I shall have to tax the reader's patience enough anyhow.

PART I. ALGEBRA. NUMBER THEORY. GROUPS.

2. Rings, Fields, Ideals. Indeed, at this point it seems impossible for me to go on without illustrating the axiomatic approach by some of the simplest algebraic notions. Some of them are as old as Methuselah. For what is older than the sequence of *natural numbers* 1, 2, 3, $\cdots$, by which we count? Two such numbers a, b may be added and multiplied ($a + b$ and $a \cdot b$). The next step in the genesis of numbers adds to these positive *integers* the negative ones and zero; in the wider system thus created the operation of addition permits of a unique inversion, subtraction. One does not stop here: the integers in their turn get absorbed into the still wider range of *rational numbers* (fractions). Thereby division, the operation inverse to multiplication, also becomes possible, with one notable exception however: division by zero. (Since $b \cdot 0 = 0$ for every rational number b, there is no inverse b of 0 such that $b \cdot 0 = 1$.) I now formulate the fundamental facts about the operations "plus" and "times" in the form of a table of axioms:

Table T

(1) The commutative and associative laws for addition,

$$a + b = b + a. \qquad a + (b + c) = (a + b) + c.$$

(2) The corresponding laws for multiplication.

(3) The distributive law connecting addition with multiplication

$$c \cdot (a + b) = (c \cdot a) + (c \cdot b).$$

(4) The axioms of subtraction: (4_1) There is an element o (0, "zero") such that $a + o = o + a = a$ for every a. (4_2) To every a there is a number $-a$ such that $a + (-a) = (-a) + a = o$.

(5) The axioms of division: (5_1) There is an element e (1, "unity") such that $a \cdot e = e \cdot a = a$ for every a. (5_2) To every $a \neq o$ there is an a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

By means of (4_2) and (5_2) one may introduce the difference $b - a$ and the quotient b/a as $b + (-a)$ and $b \cdot a^{-1}$, respectively.

When the Greeks discovered that the ratio $(\sqrt{2})$ between diagonal and side of a square is not measurable by a rational number, a further extension of the number concept was called for. However, all measurements of continuous quantities are possible only approximately, and always have a certain range of inaccuracy. Hence rational numbers, or even finite decimal fractions, can and do serve the ends of mensuration provided they are interpreted as approximations, and a calculus with approximate numbers seems the adequate numerical instrument for all measuring sciences. But mathematics ought to be prepared for any subsequent refinement of measurements. Hence dealing, say, with electric phenomena, one would be glad if one could consider the approximate values of the charge e of the electron which the experimentalist determines with ever greater accuracy as approximations of one definite *exact* value e. And thus, during more than two millenniums from Plato's time until the end of the nineteenth century, the mathematicians worked out an exact number concept, that of *real numbers*, that underlies all our theories in natural science. Not even to this day are the logical issues involved in that concept completely clarified and settled. The rational numbers are but a small part of the real numbers. The latter satisfy our axioms no less than the rational ones, but their system possesses a certain completeness not enjoyed by the rational numbers, and it is this, their "topological" feature, on which the operations with infinite sums and the like, as well as all continuity arguments, rest. We shall come back to this later.

Finally, during the Renaissance *complex numbers* were introduced. They are essentially pairs $z = (x, y)$ of real numbers x, y, pairs for which addition and multiplication are defined in such a way that all axioms hold. On the ground of these definitions $e = (1, 0)$ turns out to be the unity, while $i = (0, 1)$ satisfies the equation $i \cdot i = -e$. The two members x, y of the pair z are called its real and imaginary parts, and z is usually written in the form $xe + yi$, or simply $x + yi$. The usefulness of the complex numbers rests on the fact that every algebraic equation (with real or complex coefficients) is solvable in the field of complex numbers. The analytic functions of a complex variable are the subject of a particularly rich and harmonious theory, which is the show-piece of classical nineteenth century analysis.

A set of elements for which the operations $a + b$ and $a \cdot b$ are so defined as to satisfy the axioms (1)–(4) is called a *ring*; it is called a *field* if also the axioms (5) hold. Thus the common integers form a ring I, the rational numbers form a field ω; so do the real numbers (field Ω) and the complex numbers (field Ω^*). But these are by no means the only rings or fields. The polynomials of all

possible degrees h,

$$f = f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_hx^h, \tag{1}$$

with coefficients a_i taken from a given ring R (*e.g.* the ring I of integers, or the field ω), called "polynomials over R," form a ring $R[x]$. Here the variable or indeterminate x is to be looked upon as an empty symbol; the polynomial is really nothing but the sequence of its coefficients $a_0, a_1, a_2, \cdots$. But writing it in the customary form (1) suggests the rules for the addition and multiplication of polynomials which I will not repeat here. By substituting for the variable x a definite element ("number") γ of R, or of a ring P containing R as a subring, one projects the elements f of $R[x]$ into elements α of P, $f \to \alpha$: the polynomial $f = f(x)$ goes over into the number $\alpha = f(\gamma)$. This mapping $f \to \alpha$ is ***homomorphic***, ***i.e.***, it preserves addition and multiplication. Indeed, if the substitution of γ for x carries the polynomial f into α and the polynomial g into β then it carries $f + g, f \cdot g$ into $\alpha + \beta, \alpha \cdot \beta$, respectively.

If the product of two elements of a ring is never zero unless one of the factors is, one says that the ring is without null-divisor. This is the case for the rings discussed so far. A field is always a ring without null-divisor. The construction by which one rises from the integers to the fractions can be used to show that any ring R with unity and without null-divisor may be imbedded in a field k, the quotient field, such that every element of k is the quotient a/b of two elements a and b of R, the second of which (the denominator) is not zero.

Writing $1a, 2a, 3a, \cdots$, for $a, a + a, a + a + a$, *etc.*, we use the natural numbers $n = 1, 2, 3, \cdots$, as multipliers for the elements a of a ring or a field. Suppose the ring contains the unity e. It may happen that a certain multiple ne of e equals zero; then one readily sees that $na = 0$ for every element a of the ring. If the ring is without null divisors, in particular if it is a field and p is the least natural number for which $pe = 0$, then p is necessarily a prime number like 2 or 3 or 5 or 7 or 11 $\cdots$. One thus distinguishes fields of prime characteristic p from those of characteristic 0 in which no multiple of e is zero.

Plot the integers $\cdots, -2, -1, 0, 1, 2, \cdots$ as equidistant marks on a line. Let n be a natural number $\geqq 2$ and roll this line upon a wheel of circumference n. Then any two marks a, a' coincide, the difference $a - a'$ of which is divisible by n. (The mathematicians write $a \equiv a' \pmod{n}$; they say: a congruent to a' modulo n.) By this identification the ring of integers I goes over into a ring I_n consisting of n elements only (the marks on the wheel), as which one may take the "residues" 0, 1, $\cdots$, $n - 1$. Indeed, congruent numbers give congruent results under both addition and multiplication: $a \equiv a', b \equiv b' \pmod{n}$ imply $a + b \equiv a' + b', a \cdot b \equiv a' \cdot b' \pmod{n}$. For instance, modulo 12 we have $7 + 8 = 3, 5 \cdot 8 = 4$ because 15 leaves the residue 3 and 40 the residue 4 if divided by 12. The ring I_{12} is not without null divisors since $3 \cdot 4$ is divisible by 12, but neither 3 nor 4 is. However, if p is a natural prime number, then I_p has no null divisor and is even a field; for as the ancient Greeks proved by an ingenious procedure (Euclid's algorism), for every integer a not divisible by

p there is one, a', such that $a \cdot a' \equiv 1 \pmod{p}$. This Euclidean theorem is at the basis of the whole of number theory. The example shows that there are fields of any given prime characteristic p.

In any ring R one may introduce the notions of unit and prime element as follows. The ring element a is a unit if it has a reciprocal a' in the ring, such that $a' \cdot a = e$. The element a is composite if it may be decomposed into two factors $a_1 \cdot a_2$, neither of which is a unit. A prime number is one that is neither a unit nor composite. The units of I are the numbers $+1$ and -1. The units of the ring $k[x]$ of polynomials over a field k are the non-vanishing elements of k (polynomials of degree 0). According to the Greek discovery of the irrationality of $\sqrt{2}$ the polynomial $x^2 - 2$ is prime in the ring $\omega[x]$; but, of course, not in $\Omega[x]$, for there it splits into the two linear factors $(x - \sqrt{2})(x + \sqrt{2})$. Euclid's algorism is also applicable to polynomials $f(x)$ of one variable x over any field k. Hence they satisfy Euclid's theorem: Given a prime element $P = P(x)$ in this ring $k[x]$ and an element $f(x)$ of $k[x]$ not divisible by $P(x)$, there exists another polynomial $f'(x)$ over k such that $\{f(x) \cdot f'(x)\} - 1$ is divisible by $P(x)$. Identification of any elements f and g of $k[x]$, the difference of which is divisible by P, therefore changes the ring $k[x]$ into a field, the "residue field κ of $k[x]$ modulo P." Example: $\omega[x]$ mod $x^2 - 2$. (Incidentally the complex numbers may be described as the elements of the residue field of $\Omega[x]$ mod $x^2 + 1$.) Strangely enough, the fundamental Euclidean theorem does not hold for polynomials of two variables x, y. For instance, $P(x, y) = x - y$ is a prime element of $\omega[x, y]$, and $f(x, y) = x$ an element not divisible by $P(x, y)$. But a congruence

$$x \cdot f'(x, y) \equiv 1 \pmod{x - y}$$

is impossible. Indeed, it would imply $-1 + x \cdot f'(x, x) = 0$, contrary to the fact that the polynomial of one indeterminate x,

$$-1 + x \cdot f'(x, x) = -1 + c_1 x + c_2 x^2 + \cdots,$$

is not zero. Thus the ring $\omega[x, y]$ does not obey the simple laws prevailing in I and in $\omega[x]$.

Consider κ, the residue field of $\omega[x]$ mod $x^2 - 2$. Since for any two polynomials $f(x)$, $f'(x)$ which are congruent mod $x^2 - 2$ the numbers $f(\sqrt{2})$, $f'(\sqrt{2})$ coincide, the transition $f(x) \to f(\sqrt{2})$ maps κ into a sub-field $\omega[\sqrt{2}]$ of Ω consisting of the numbers $a + b\sqrt{2}$ with rational a, b. Another such projection would be $f(x) \to f(-\sqrt{2})$. In former times one looked upon κ as the part $\omega[\sqrt{2}]$ of the continuum Ω or Ω^* of all real or all complex numbers; one wished to embed everything into this universe Ω or Ω^* in which analysis and physics operate. But as we have introduced it here, κ is an algebraic entity the elements of which are not numbers in the ordinary sense. It requires for its construction no other numbers but the rational ones. It has nothing to do with Ω, and ought not to be confused with the one or the other of its two projections into Ω. More generally, if $P = P(x)$ is any prime element in $\omega[x]$ we can form the

residue field κ_P of $\omega[x]$ modulo P. To be sure, if δ is any of the real or complex roots of the equation $P(x)=0$ in Ω^* then $f(x) \rightarrow f(\delta)$ defines a homomorphic projection of κ_P into Ω^*. But the projection is not κ_P itself.

Let us return to the ordinary integers $\cdots, -2, -1, 0, 1, 2, \cdots$, which form the ring I. The multiples of 5, *i.e.*, the integers divisible by 5, clearly form a ring. It is a ring without unity, but it has another important peculiarity: not only does the product of any two of its elements lie in it, but all the integral multiples of an element do. The queer term *ideal* has been introduced for such a set: Given a ring R, an R-ideal $(\mathfrak{a})$ is a set of elements of R such that (1) sum and difference of any two elements of $(\mathfrak{a})$ are in $(\mathfrak{a})$, (2) the product of an element in $(\mathfrak{a})$ by any element of R is in $(\mathfrak{a})$. We may try to describe a divisor $\mathfrak{a}$ by the set of all elements divisible by $\mathfrak{a}$. One would certainly expect this set to be an ideal $(\mathfrak{a})$ in the sense just defined. Given an ideal $(\mathfrak{a})$, there may not exist an actual element a of R such that $(\mathfrak{a})$ consists of all multiples $j = m \cdot a$ of a (m any element in R). But then we would say that $(\mathfrak{a})$ stands for an "*ideal* divisor" $\mathfrak{a}$: the words "the element j of R is divisible by $\mathfrak{a}$" would simply mean: "j belongs to $(\mathfrak{a})$." In the ring I of common integers all divisors are actual.

But this is not so in every ring. An algebraic surface in the three-dimensional Euclidean space with the Cartesian coordinates x, y, z is defined by an equation $F(x, y, z) = 0$ where F is an element of ${}^3\Omega = \Omega[x, y, z]$, *i.e.*, a polynomial of the variables x, y, z with real coefficients. F is zero in all the points of the surface; but the same is true for every multiple $L \cdot F$ of F (L being any element of ${}^3\Omega$), in other words, for every polynomial of the ideal (F) in ${}^3\Omega$. Two simultaneous polynomial equations

$$F_1(x, y, z) = 0, \qquad F_2(x, y, z) = 0$$

will in general define a curve, the intersection of the surface $F_1 = 0$ and the surface $F_2 = 0$. The polynomials $(L_1 \cdot F_1) + (L_2 \cdot F_2)$ formed by arbitrary elements L_1, L_2 of ${}^3\Omega$ form an ideal (F_1, F_2), and all these polynomials vanish on the curve. This ideal will in general not correspond to an actual divisor F, for a curve is not a surface. Examples like this should convince the reader that the study of algebraic manifolds (curves, surfaces, *etc.*, in 2, 3, or any number of dimensions) amounts essentially to a study of polynomial ideals. The field of coefficients is not necessarily Ω or Ω^*, but may be a field of a more general nature.

3. Some achievements of algebra and number theory. I have finally reached a point where I can hint, I hope, with something less than complete obscurity, at some of the accomplishments of algebra and number theory in our century. The most important is probably the freedom with which we have learned to manage these abstract axiomatic concepts, like field, ring, ideal, *etc.* The atmosphere in a book like van der Waerden's *Moderne Algebra*, published about 1930, is completely different from that prevailing, *e.g.*, in the articles on algebra written for the *Mathematical Encyclopaedia* around 1900. More specif-

ically, a general theory of ideals, and in particular of polynomial ideals, was developed. (However, it should be said that the great pioneer of abstract algebra, Richard Dedekind, who first introduced the ideals into number theory, still belonged to the nineteenth century.) Algebraic geometry, before and around 1900 flourishing chiefly in Italy, was at that time a discipline of a type uncommon in the sisterhood of mathematical disciplines: it had powerful methods, plenty of general results, but they were of somewhat doubtful validity. By the abstract algebraic methods of the twentieth century all this was put on a safe basis, and the whole subject received a new impetus. Admission of fields other than Ω^*, as the field of coefficients, opened up a new horizon.

A new technique, the "primadic numbers," was introduced into algebra and number theory by K. Hensel shortly after the turn of the century, and since then has become of ever increasing importance. Hensel shaped this instrument in analogy to the power series which played such an important part in Riemann's and Weierstrass's theory of algebraic functions of one variable and their integrals (Abelian integrals). In this theory, one of the most impressive accomplishments of the previous century, the coefficients were supposed to vary over the field Ω^* of all complex numbers. Without pursuing the analogy, I may illustrate the idea of p-adic numbers by one typical example, that of quadratic norms. Let p be a prime number, and let us first agree that a congruence $a \equiv b$ modulo a power p^λ of p for rational numbers a, b has this meaning that $(a - b)/p^\lambda$ equals a fraction whose denominator is not divisible by p;

$$e.g., \frac{39}{4} - \frac{12}{5} \equiv 0 \pmod{7^2} \text{ because } \frac{39}{4} - \frac{12}{5} = 7^2 \cdot \frac{3}{20}.$$

Let now a, b be rational numbers, $a \neq 0$, and b not the square of a rational number. In the quadratic field $\omega[\sqrt{b}]$ the number a is a *norm* if there are rational numbers x, y such that

$$a = (x + y\sqrt{b})(x - y\sqrt{b}), \qquad \text{or} \qquad a = x^2 - by^2.$$

Necessary for the solvability of this equation is (1) that for every prime p and every power p^λ of p the congruence $a \equiv x^2 - by^2 \pmod{p^\lambda}$ has a solution. This is what we mean by saying the equation has a p-adic solution. Moreover there must exist rational numbers x and y such that $x^2 - by^2$ differs as little as one wants from a. This is what we mean by saying that the equation has an ∞-adic solution. The latter condition is clearly satisfied for every a provided b is positive; however, if b is negative it is satisfied only for positive a. In the first case every a is ∞-adic norm, in the second case only half of the a's are, namely, the positive ones. A similar situation prevails with respect to p-adic norms. One proves that these necessary conditions are also sufficient: if a is a norm locally everywhere, *i.e.*, if $a = x^2 - by^2$ has a p-adic solution for every "finite prime spot p" and also for the "infinite prime spot ∞," then it has a "global" solution, namely an exact solution in rational numbers x, y.

This example, the simplest I could think of, is closely connected with the theory of genera of quadratic forms, a subject that goes back to Gauss' *Disquisitiones arithmeticae*, but in which the twentieth century has made some decisive progress by means of the p-adic technique, and it is also typical for that most fascinating branch of mathematics mentioned in the introduction: class field theory. Around 1900 David Hilbert had formulated a number of interlaced theorems concerning class fields, proved some of them at least in special cases, and left the rest to his twentieth century successors, among whom I name Takagi, Artin and Chevalley. His norm residue symbol paved the way for Artin's general reciprocity law. Hilbert had used the analogy with the Riemann-Weierstrass theory of algebraic functions over Ω^* for his orientation, but the ingenious, partly transcendental methods which he applied had nothing to do with the much simpler ones that had proved effective for the functions. By the primadic technique a rapprochement of methods has occurred, although there is still a considerable gap separating the theory of algebraic functions and the much subtler algebraic numbers.

Hensel and his successors have expressed the p-adic technique in terms of the non-algebraic "topological" notion of ("valuation" or) *convergence*. An infinite sequence of rational numbers $a_1, a_2, \cdots$ is convergent if the difference $a_i - a_j$ tends to zero, $a_i - a_j \to 0$, provided i and j independently of each other tend to infinity; more explicitly, if for every positive rational number ϵ there exists a positive integral N such that $-\epsilon < a_i - a_j < \epsilon$ for all i and $j > N$. The completeness of the real number system is expressed by Cauchy's convergence theorem: To every convergent sequence $a_1, a_2, \cdots$ of rational numbers there exists a *real* number α to which it converges: $a_i - \alpha \to 0$ for $i \to \infty$. With the ∞-adic concept of convergence we have now confronted the p-adic one induced by a prime number p. Here the sequence is considered convergent if for every exponent $h = 1, 2, 3, \cdots$, there is a positive integer N such that $a_i - a_j$ is divisible by p^h as soon as i and $j > N$. By introduction of p-adic numbers one can make the system of rational numbers complete in the p-adic sense as the introduction of real numbers makes them complete in the ∞-adic sense. The rational numbers are embedded in the continuum of all real numbers, but they may be embedded as well in that of all p-adic numbers. Each of these embedments corresponding to a finite or the infinite prime spot p is equally interesting from the arithmetical viewpoint. Now it is more evident than ever how wrong it was to identify an algebraic number field with one of its homomorphic projections into the field Ω of real numbers; along with the (real) infinite prime spots one must pay attention to the finite prime spots which correspond to the various prime ideals of the field. This is a golden rule abstracted from earlier, and then made fruitful for later, arithmetical research; and here is one bridge (others will be pointed out later) joining the two most fascinating branches of modern mathematics: abstract algebra and topology.

Besides the introduction of the primadic treatment and the progress made in the theory of class fields, the most important advances of number theory

during the last fifty years seem to lie in those regions where the powerful tool of analytic functions can be brought to bear upon its problems. I mention two such fields of investigation: I. distribution of primes and the zeta function, II. additive number theory.

I. The notion of prime number is of course as old and as primitive as that of the multiplication of natural numbers. Hence it is most surprising to find the distribution of primes among all natural numbers is of such a highly irregular and almost mysterious character. While on the whole the prime numbers thin out the further one gets in the sequence of numbers, wide gaps are always followed again by clusters. An old conjecture of Goldbach's maintains that there even come along again and again pairs of primes of the smallest possible difference 2, like 57 and 59. However, the distribution of primes obeys at least a fairly simple *asymptotic* law: the number $\pi(n)$ of primes among all numbers from 1 to n is asymptotically equal to $n/\log n$. [Here $\log n$ is not the Briggs logarithm which our logarithmic tables give, but the natural logarithm as defined by the integral $\int_1^n dx/x$.] By asymptotic is meant that the quotient between $\pi(n)$ and the approximating function $n/\log n$ tends to 1 as n tends to infinity. In antiquity Eratosthenes had devised a method to sift out the prime numbers. By this sieve method the Russian mathematician Tchebycheff had obtained, during the nineteenth century, the first non-trivial results about the distribution of primes. Riemann used a different approach: his tool is the so-called zeta-function defined by the infinite series

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \cdots. \tag{2}$$

Here s is a complex variable, and the series converges for all values of s, the real part of which is greater than 1, $\Re s > 1$. Already in the eighteenth century the fact that every positive integer can be uniquely factorized into primes had been translated by Euler into the equation

$$1/\zeta(s) = (1 - 2^{-s})(1 - 3^{-s})(1 - 5^{-s}) \cdots$$

where the (infinite) product extends over all primes 2, 3, 5, $\cdots$. Riemann showed that the zeta-function has a unique "analytic continuation" to all values of s and that it satisfies a certain functional equation connecting its values for s and $1 - s$. Decisive for the prime number problem are the zeros of the zeta-function, *i.e.*, the values s for which $\zeta(s)=0$. Riemann's equation showed that, except for the "trivial" zeros at $s = -2, -4, -6, \cdots$, all zeros have real parts between 0 and 1. Riemann conjectured that their real parts actually equal $\frac{1}{2}$. His conjecture has remained a challenge to mathematics now for almost a hundred years. However, enough had been learned about these zeros at the close of the nineteenth century to enable mathematicians, by means of some profound and newly-discovered theorems concerning analytic functions, to prove the above-mentioned asymptotic law. This was generally considered a great triumph of mathematics. Since the turn of the century Rie-

mann's functional equation with the attending consequences has been carried over from the "classical" zeta-function (ii) of the field of rational numbers to that of an arbitrary algebraic number field (E. Hecke). For certain fields of prime characteristic one succeeded in confirming Riemann's conjecture, but this provides hardly a clue for the classical case. About the classical zeta-function we know now that it has infinitely many zeros on the critical line $\Re s = \frac{1}{2}$, and even that at least a fixed percentage, say 10 per cent, of them lie on it. (What this means is the following: Some percentage of those zeros whose imaginary part lies between arbitrary fixed limits $-T$ and $+T$ will have a real part equal to $\frac{1}{2}$, and this percentage will not sink below a certain positive limit, like 10 per cent, when T tends to infinity.) Finally about two years ago Atle Selberg succeeded, to the astonishment of the mathematical world, in giving an "elementary" proof of the prime number law by an ingenious refinement of old Eratosthenes' sieve method.

II. It has been known for a long time that every natural number n may be written as the sum of at most four square numbers, *e.g.*,

$$7 = 2^2 + 1^2 + 1^2 + 1^2, \qquad 87 = 9^2 + 2^2 + 1^2 + 1^2 = 7^2 + 5^2 + 3^2 + 2^2.$$

The same question arises for cubes, and generally for any k^{th} powers ($k = 2, 3, 4, 5, \cdots$). In the eighteenth century Waring had conjectured that every non-negative integer n may be expressed as the sum of a limited number M of k^{th} powers,

$$n = n_1^k + n_2^k + \cdots + n_M^k, \tag{3}$$

where the n_i are also non-negative integers and M is independent of n. The first decade of the twentieth century brought two events: first one found that every n is expressible as the sum of at most 9 cubes (and that, excepting a few comparatively small n, even 8 cubes will do); and shortly afterwards Hilbert proved Waring's general theorem. His method was soon replaced by a different approach, the Hardy-Littlewood circle method, which rests on the use of a certain analytic function of a complex variable and yields asymptotic formulas for the number of different representations of n in the form (3). With some precautions demanded by the nature of the problem, and by overcoming some quite serious obstacles, the result was later carried over to arbitrary algebraic number fields; and by a further refinement of the circle method in a different direction Vinogradoff proved that every sufficiently large n is the sum of at most 3 primes. Is it even true that every even n is the sum of 2 primes? To show this seems to transcend our present mathematical powers as much as Goldbach's conjecture. The prime numbers remain very elusive fellows.

III. Finally, a word ought to be said about investigations concerning the arithmetical nature of numbers originating in analysis. One of the most elementary such constants is π, the area of the circle of radius 1. By proving that π is a transcendental number (not satisfying an algebraic equation with

rational coefficients) the age-old problem of "squaring the circle" was settled in 1882 in the negative sense; that is, one cannot square the circle by constructions with ruler and compass. In general it is much harder to establish the transcendency of numbers than of functions. Whereas it is easy to see that the exponential function

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \cdots$$

is not algebraic, it is quite difficult to prove that its basis e is a transcendental number. C. L. Siegel was the first who succeeded, around 1930, in developing a sort of general method for testing the transcendency of numbers. But the results in this field remain sporadic.

4. Groups, vector spaces and algebras. This ends our report on number theory, but not on algebra. For now we have to introduce the *group* concept, which, since the young genius Evariste Galois blazed the trail in 1830, has penetrated the entire body of mathematics. Without it an understanding of modern mathematics is impossible. Groups first occurred as *groups of transformations*. Transformations may operate in any set of elements, whether it is finite like the integers from 1 to 10, or infinite like the points in space. *Set* is a pre-mathematical concept: whenever we deal with a realm of objects, a set is defined by giving a criterion which decides for any object of the realm whether it belongs to the set or not. Thus we speak of the set of prime numbers, or of the set of all points on a circle, or of all points with rational coordinates in a given coordinate system, or of all people living at this moment in the State of New Jersey. Two sets are considered equal if every element of the one belongs to the other and vice versa. A *mapping* S of a set σ into a set σ' is defined if with every element a of σ there is associated an element a' of σ', $a \to a'$. Here a rule is required which allows one to find the "image" a' for any given element a of σ. This general notion of mapping we may also call of a pre-mathematical nature. Examples: a real-valued function of a real variable is a mapping of the continuum Ω into itself. Perpendicular projection of the space points upon a given plane is a mapping of the space into the plane. Representing every space-point by its three coordinates x, y, z with respect to a given coordinate system is a mapping of space into the continuum of real number triples (x, y, z). If a mapping S, $a \to a'$ of σ into σ', is followed by a mapping S', $a' \to a''$ of σ' into a third set σ'', the result is a mapping SS': $a \to a''$ of σ into σ''. A *one-to-one mapping* between two sets σ, σ' is a pair of mappings, S: $a \to a'$ of σ into σ', and S': $a' \to a$ of σ' into σ, which are inverse to each other. This means that the mapping SS' of σ into σ is the identical mapping E of σ which sends every element a of σ into itself, and that $S'S$ is the identical mapping of σ'. In particular, one is interested in one-to-one mappings of a set σ into itself. For them we shall use the word *transformation*. Permutations are nothing but transformations of a finite set.

The inverse S' of a transformation S, $a \rightarrow a'$ of a given set σ, is again a transformation and is usually denoted by S^{-1}. The result ST of any two transformations S and T of σ is again a transformation, and its inverse is $T^{-1}S^{-1}$ (according to the rule of dressing and undressing: if in dressing one begins with the shirt and ends with the jacket, one must in undressing begin with the jacket and end with the shirt. The order of the two "factors" S, T is essential.) *A group of transformations* is a set of transformations of a given manifold which (1) contains the identity E, (2) contains with every transformation S its inverse S^{-1}, and (3) with any two transformations S, T their "product" ST. Example: One could define congruent configurations in space as point sets of which one goes into the other by a congruent transformation of space. The congruent transformations, or "motions," of space form a group; a statement which, according to the above definition of group, is equivalent to the threefold statement that (1) every figure is congruent to itself, (2) if a figure F is congruent to F', then F' is congruent to F, and (3) if F is congruent to F' and F' congruent to F'', then F is congruent to F''. This example at once illuminates the inner significance of the group concept. *Symmetry* of a configuration F in space is described by the group of motions that carry F into itself.

Often manifolds have a structure. For instance, the elements of a field are connected by the two operations of plus and times; or in Euclidean space we have the relationship of congruence between figures. Hence we have the idea of structure-preserving mappings; they are called *homomorphisms*. Thus a homomorphic mapping of a field k into a field k' is a mapping $a \rightarrow a'$ of the "numbers" a of k into the numbers a' of k' such that $(a + b)' = a' + b'$ and $(a \cdot b)' = a' \cdot b'$. A homomorphic mapping of space into itself would be one that carries any two congruent figures into two mutually congruent figures. The following terminology (suggestive to him who knows a little Greek) has been agreed upon: homomorphisms which are one-to-one mappings are called isomorphisms; when a homomorphism maps a manifold σ into itself, it is called an endomorphism, and an automorphism when it is both: a one-to-one mapping of σ into itself. Isomorphic systems, *i.e.*, any two systems mapped isomorphically upon each other, have the same structure; indeed nothing can be said about the structure of the one system that is not equally true for the other.

The *automorphisms* of a manifold with a well-defined structure form a *group*. Two sub-sets of the manifold that go over into each other by an automorphism deserve the name of *equivalent*. This is the precise idea at which Leibniz hints when he says that two such sub-sets are "indiscernible when each is considered in itself"; he recognized this general idea as lying behind the specific geometric notion of similitude. The general problem of relativity consists in nothing else but to find the group of automorphisms. Here then is an important lesson the mathematicians learned in the twentieth century: whenever you are concerned with a structured manifold, study its group of automorphisms. Also the inverse problem, which Felix Klein stressed in his famous Erlangen program (1872), deserves attention: Given a group of transformations of a manifold σ,

determine such relations or operations as are invariant with respect to the group.

If in studying a group of transformations we ignore the fact that it consists of transformations and look merely at the way in which any two of its transformations S, T give rise to a composite ST, we obtain the abstract composition schema of the group. Hence an *abstract group* is a set of elements (of unknown or irrelevant nature) for which an operation of composition is defined generating an element st from any two elements s, t such that the following axioms hold:

(1) There is a unit element e such that $es=se=s$ for every s.

(2) Every element s has an inverse s^{-1} such that $ss^{-1} = s^{-1}s = e$.

(3) The associative law $(st)u = s(tu)$ holds.

It is perhaps the most astonishing experience of modern mathematics how rich in consequences these three simple axioms are. A realization of an abstract group by transformations of a given manifold σ is obtained by associating with every element s of the group a transformation S of σ, $s \rightarrow S$, such that $s \rightarrow S$, $t \rightarrow T$ imply $st \rightarrow ST$. In general, the commutative law $st = ts$ will not hold. If it does, the group is called commutative or Abelian (after the Norwegian mathematician Niels Henrik Abel). Because composition of group elements in general does not satisfy the commutative law, it has proved convenient to use the term "ring" in the wider sense in which it does not imply the commutative law for multiplication. (However, in speaking of a field one usually assumes this law.)

The simplest mappings are the linear ones. They operate in a vector space. The vectors in our ordinary three-dimensional space are directed segments AB leading from a point A to a point B. The vector AB is considered equal to $A'B'$ if a parallel displacement (translation) carries AB into $A'B'$. In consequence of this convention one can add vectors and one can also multiply a vector by a number (integral, rational or even real). Addition satisfies the same axioms as enumerated for numbers in the table **T**, and it is also easy to formulate the axioms for the second operation. These axioms constitute the general axiomatic notion of vector space, which is therefore an algebraic and not a geometric concept. The numbers which serve as multipliers of the vectors may be the elements of any ring; this generality is actually required in the application of the axiomatic vector concept to topology. However, here we shall assume that they form a field. Then one sees at once that one can ascribe to the vector space a natural number n as its dimensionality in this sense: there exist n vectors $e_1, \cdots, e_n$ such that every vector may be expressed in one and only one way as a linear combination $x_1e_1 + \cdots + x_ne_n$, where the "coordinates" x_i are definite numbers of the field. In our three-dimensional space n equals 3, but mechanics and physics give ample occasion to use the general algebraic notion of an n-dimensional vector space for higher n.

The endomorphisms of a vector space are called its *linear mappings*; as such they allow composition ST (perform first the mapping S, then T), but they also allow addition and multiplication by numbers γ: if S sends the

arbitrary vector x into xS, T into xT, then $S + T$, γS are those linear mappings which send x into $(xS) + (xT)$ and $\gamma \cdot xS$, respectively. We must forego to describe how in terms of a vector basis $e_1, \cdots, e_n$ a linear mapping is represented by a square matrix of numbers.

Often rings occur—they are then called *algebras*—which are at the same time vector spaces, *i.e.*, for which three operations, addition of two elements, multiplication of two elements and multiplication of an element by a number, are defined in such manner as to satisfy the characteristic axioms. The linear mappings of an n-dimensional vector space themselves form such an algebra, called the complete matric algebra (in n dimensions). According to quantum mechanics the observables of a physical system form an algebra of special type with a non-commutative multiplication. In the hands of the physicists abstract algebra has thus become a key that unlocked to them the secrets of the atom. A realization of an abstract group by linear transformations of a vector space is called *representation*. One may also speak of representations of a ring or an algebra: in each case the representation can be described as a homomorphic mapping of the group or ring or algebra into the complete matric algebra (which indeed is a group and a ring and an algebra, all in one).

5. Finale. After spending so much time on the explanation of the notions I can be brief in my enumeration of some of the essential achievements for which they provided the tools. If g is a subgroup of the group G, one may identify elements s, t of G that are congruent mod g, *i.e.*, for which st^{-1} is in g; g is a "self-conjugate" subgroup if this process of identification carries G again into a group, the "factor group" G/g. The group-theoretic core of Galois' theory is a theorem due to C. Jordan and O. Hölder which deals with the several ways in which one may break down a given finite group G in steps $G = G_0, G_1, G_2, \cdots$, each G_i being a self-conjugate subgroup of the preceding group G_{i-1}. Under the assumption that this is done in as small steps as possible, the theorem states, the steps (factor groups) G_{i-1}/G_i $(i = 1, 2, \cdots)$ in one such "composition series" are isomorphic to the steps, suitably rearranged, in a second such series. The theorem is very remarkable in itself, but perhaps the more so as its proof rests on the same argument by which one proves what I consider the most fundamental proposition in all mathematics, namely the fact that if you count a finite set of elements in two ways, you end up with the same number n both times. The Jordan-Hölder theorem in recent times received a much more natural and general formulation by (1) abandoning the assumption that the breaking down is done in the smallest possible steps, and (2) by admitting only such subgroups as are invariant with respect to a given set of endomorphic mappings of G. It thus has become applicable to infinite as well as finite groups, and provided a common denominator for quite a number of important algebraic facts.

The theory of representations of finite groups, the most systematic and substantial part of group theory developed shortly before the turn of the century

by G. Frobenius, taught us that there are only a few irreducible representations, of which all others are composed. This theory was greatly simplified after 1900 and later carried over, first to continuous groups that have the topological property of compactness, but then also to all infinite groups, with a restrictive imposition (called almost-periodicity) on the representations. With these generalizations one trespasses the limits of algebra, and a few more words will have to be said about it under the title analysis. New phenomena occur if representations of finite groups in fields of prime characteristic are taken into account, and from their investigation profound number-theoretic consequences have been derived. It is easy to embed a finite group into an algebra, and hence facts about representations of a group are best deduced from those of the embedding algebra. At the beginning of the century algebras seemed to be ferocious beasts of unpredictable behavior, but after fifty years of investigation they, or at least the variety called semi-simple, have become remarkably tame; indeed the wild things do not happen in these superstructures, but in the underlying commutative "number" fields. In the nineteenth century geometry seemed to have been reduced to a study of invariants of groups; Felix Klein formulated this standpoint explicitly in his Erlangen program. But the full linear group was practically the only group whose invariants were studied. We have now outgrown this limitation and no longer ignore all the other continuous groups one encounters in algebra, analysis, geometry and physics. Above all we have come to realize that the theory of invariants has to be subsumed under that of representations. Certain infinite discontinuous groups, like the unimodular and the modular groups, which are of special importance to number theory, witness Gauss' class theory of quadratic forms, have been studied with remarkable success and profound results. The macroscopic and microscopic symmetries of crystals are described by discontinuous groups of motions, and it has been proved for n dimensions, what had long been known for 3 dimensions, that in a certain sense there is but a finite number of possibilities for these crystallographic groups. In the nineteenth century Sophus Lie had reduced a continuous group to its "germ" of infinitesimal elements. These elements form a sort of algebra in which the associative law is replaced by a different type of law. A Lie algebra is a purely algebraic structure, especially if the numbers which act as multipliers are taken from an algebraically defined field rather than from the continuum of real numbers Ω. These Lie groups have provided a new playground for our algebraists.

The constructions of the mathematical mind are at the same time free and necessary. The individual mathematician feels free to define his notions and to set up his axioms as he pleases. But the question is, will he get his fellow-mathematicians interested in the constructs of his imagination. We can not help feeling that certain mathematical structures which have evolved through the combined efforts of the mathematical community bear the stamp of a necessity not affected by the accidents of their historical birth. Everybody who looks

at the spectacle of modern algebra will be struck by this complementarity of freedom and necessity.

Part II. Analysis. Topology. Geometry. Foundations.

6. Linear operators and their spectral decomposition. Hilbert space. A mechanical system of n degrees of freedom in stable equilibrium is capable of oscillations deviating "infinitely little" from the state of equilibrium. It is a fact of fundamental significance not only for physics but also for music that all these oscillations are superpositions of n "harmonic" oscillations with definite frequencies. Mathematically the problem of determining the harmonic oscillations amounts to constructing the principal axes of an ellipsoid in an n-dimensional Euclidean space. Representing the vectors x in this space by their coordinates $(x_1, x_2, \cdots, x_n)$ one has to solve an equation

$$x - \lambda \cdot Kx = 0,$$

where K denotes a given linear operator (= linear mapping); λ is the square of the unknown frequency ν of the harmonic oscillation, whereas the "eigenvector" x characterizes its amplitude. Define the scalar product (x, y) of two vectors x and y by the sum $x_1y_1 + \cdots + x_ny_n$. Our "affine" vector space is made into a metric one by assigning to any vector x the length $\|x\|$ given by $\|x\|^2 = (x, x)$, and this metric is the Euclidean one so familiar to us from the 3-dimensional case and epitomized by the "Pythagoras." The linear operator K is symmetric in the sense that $(x, Ky) = (Kx, y)$. The field of numbers in which we operate here is, of course, the continuum of all real numbers. Determination of the n frequencies ν or rather of the corresponding eigen-values $\lambda = \nu^2$ requires the solution of an algebraic equation of degree n (often known as the secular equation, because it first appeared in the theory of the secular perturbations of the planetary system).

More important in physics than the oscillations of a mechanical system of a finite number of degrees of freedom are the oscillations of continuous media, as the mechanical-acoustical oscillations of a string, a membrane or a 3-dimensional elastic body, and the electromagnetic-optical oscillations of the "ether." Here the vectors with which one has to operate are continuous functions $x(s)$ of a point s with one or several coordinates that vary over a given domain, and consequently K is a linear *integral* operator. Take for instance a straight string of length 1, the points of which are distinguished by a parameter s varying from 0 to 1. Here (x, x) is the integral $\int_0^1 x^2(s) \cdot ds$, and the problem of harmonic oscillations (which first suggested to the early Greeks the idea of a universe ruled by harmonious mathematical laws) takes the form of the integral equation

$$[1] \qquad x(s) - \lambda \int_0^1 K(s, t)x(t)dt = 0, \qquad (0 \leqq s \leqq 1),$$

where

[1′] $$K(s, t) = \left(\frac{a}{\pi}\right)^2 \cdot \begin{cases} s(1-t) & \text{for } s \leqq t \\ (1-s)t & \text{for } s \geqq t \end{cases},$$

and a is a constant determined by the physical conditions of the string. The solutions are

$$\lambda = (na)^2, \qquad x(s) = \sin n\pi s,$$

where n is capable of all positive integral values 1, 2, 3, $\cdots$. This fact that the frequencies of a string are integral multiples na of a ground frequency a is the basic law of musical harmony. If one prefers an optical to an acoustic language one speaks of the *spectrum* of eigen-values λ.

After Fredholm at the very close of the 19th century had developed the theory of linear integral equations it was Hilbert who in the next decade established the general *spectral theory of symmetric linear operators* K. Only twenty years earlier it had required the greatest mathematical efforts to prove the existence of the ground frequency for a membrane, and now constructive proofs for the existence of the whole series of harmonic oscillations and their characteristic frequencies were given under very general assumptions concerning the oscillating medium. This was an event of great consequence both in mathematics and theoretical physics. Soon afterwards Hilbert's approach made it possible to establish those asymptotic laws for the distribution of eigen-values the physicists had postulated in their statistical treatment of the thermodynamics of radiation and elastic bodies.

Hilbert observed that an arbitrary continuous function $x(s)$ defined in the interval $0 \leqq s \leqq 1$ may be replaced by the sequence

$$x_n = \sqrt{2} \int_0^1 x(s) \cdot \sin n\pi s \cdot ds, \qquad n = 1, 2, 3, \cdots,$$

of its Fourier coefficients. Thus there is no inner difference between a vector space whose elements are functions $x(s)$ of a continuous variable and one whose elements are infinite sequences of numbers $(x_1, x_2, x_3, \cdots)$. The square of the "length," $\int_0^1 x^2(s) \cdot ds$ equals $x_1^2 + x_2^2 + x_3^2 + \cdots$. Between the two forms in which one may pass from a finite sum to a limit, the infinite sum $a_1 + a_2 + a_3 + \cdots$ and the integral $\int_0^1 a(s) \cdot ds$, there is therefore here no essential difference. Thus an axiomatic formulation is called for. To the axioms for an (affine) vector space one adds the postulate of the existence of a scalar product (x, y) of any two vectors (x, y) with the properties characteristic for Euclidean metric: (x, y) is a number depending linearly on either of the two argument vectors x and y; it is symmetric, $(x, y) = (y, x)$; and $(x, x) = \|x\|^2$ is positive except for $x = 0$. The axiom of finite dimensionality is replaced by a denumerability axiom of more general character. All operations in such a space are greatly facilitated if it is assumed to be complete in the same sense that the system of real numbers is complete; *i.e.*, if the following is true: Given a

"convergent" sequence x', x'', $\cdots$ of vectors, namely, one for which $\|x^{(m)} - x^{(n)}\|$ tends to zero with m and n tending to infinity, there exists a vector a toward which this sequence converges, $\|x^{(n)} - a\| \to 0$ for $n \to \infty$. A non-complete vector space can be made complete by the same construction by which the system of rational numbers is completed to form that of real numbers. Later authors have coined the name "Hilbert space" for a vector space satisfying these axioms.

Hilbert himself first tackled only integral operators in the strict sense as exemplified by [1]. But soon he extended his spectral theory to a far wider class, that of bounded (symmetric) linear operators in Hilbert space. Boundedness of the linear operator requires the existence of a constant M such that $\|Kx\|^2 \leqq M \cdot \|x\|^2$ for all vectors x of finite length $\|x\|$. Indeed the restriction to integral operators would be unnatural since the simplest operator, the identity $x \to x$, is not of this type. And now one of those events happened, unforeseeable by the wildest imagination, the like of which could tempt one to believe in a pre-established harmony between physical nature and mathematical mind: Twenty years after Hilbert's investigations *quantum mechanics* found that the observables of a physical system are represented by the linear symmetric operators in a Hilbert space and that the eigen-values and eigen-vectors of that operator which represents *energy* are the energy levels and corresponding stationary quantum states of the system. Of course this quantum-physical interpretation added greatly to the interest in the theory and led to a more scrupulous investigation of it, resulting in various simplifications and extensions.

Oscillations of continua, the boundary value problems of classical physics and the problem of energy levels in quantum physics, are not the only titles for applications of the theory of integral equations and their spectra. One other somewhat isolated application is the solution of *Riemann's monodromy problem* concerning analytic functions of a complex variable z. It concerns the determination of n analytic functions of z which remain regular under analytic continuation along arbitrary paths in the z-plane provided these avoid a finite number of singular points, whereas the functions undergo a given linear transformation with constant coefficients when the path circles one of these points.

Another surprising application is to the establishment of the fundamental facts, in particular of the completeness relation, in the theory of *representations of continuous compact groups*. The simplest such group consists of the rotations of a circle, and in that case the theory of representations is nothing but the theory of the so-called Fourier series, which expresses an arbitrary periodic function $f(s)$ of period 2π in terms of the harmonic oscillations

$$\cos ns, \qquad \sin ns, \qquad n = 0, 1, 2, \cdots.$$

In Nature functions often occur with hidden non-commensurable periodicities. The mathematician Harald Bohr, the brother of the physicist Niels Bohr, prompted by certain of his investigations concerning the Riemann zeta function, developed the general mathematical theory of such *almost periodic func-*

tions. One may describe his theory as that of almost periodic representations of the simplest continuous group one can imagine, namely, the group of all translations of a straight line. His main results could be carried over to arbitrary groups. No restriction is imposed on the group, but the representations one studies are supposed to be almost-periodic. For a function $x(s)$, the argument s of which runs over the group elements, while its values are real or complex numbers, almost-periodicity amounts to the requirement that the group be compact in a certain topology induced by the function. This relative compactness instead of absolute compactness is sufficient. Even so the restriction is severe. Indeed the most important representations of the classical continuous groups are not almost-periodic. Hence the theory is in need of further extension, which has busied a number of American and Russian mathematicians during the last decade.

7. Lebesgue's integral. Measure theory. Ergodic hypothesis. Before turning to other applications of operators in Hilbert space I must mention the, in all probability final, form given to the idea of integration by Lebesgue at the beginning of our century. Instead of speaking of the area of a piece of the 2-dimensional plane referred to coordinates x, y, or the volume of a piece of the 3-dimensional Euclidean space, we use the neutral term *measure* for all dimensions. The notions of measure and *integral* are interconnected. Any piece of space, any set of space points can be described by its characteristic function $\chi(P)$, which equals 1 or 0 according to whether the point P belongs or does not belong to the set. The measure of the point set is the integral of this characteristic function. Before Lebesgue one first defined the integral for continuous functions; the notion of measure was secondary; it required transition from continuous to such discontinuous functions as $\chi(P)$. Lebesgue goes the opposite and perhaps more natural way: for him measure comes first and the integral second. The one-dimensional space is sufficient for an illustration. Consider a real-valued function $y = f(x)$ of a real variable x which maps the interval $0 \leqq x \leqq 1$ into a finite interval $a \leqq y \leqq b$. Instead of subdividing the interval of the argument x Lebesgue subdivides the interval (a, b) of the dependent variable y into a finite number of small subintervals $a_i \leqq y < a_{i+1}$, say of lengths $< \epsilon$, and then determines the measure m_i of the set S_i on the x-axis, the points of which satisfy the inequality $a_i \leqq f(x) < a_{i+1}$. The integral lies between the two sums $\Sigma_i a_i m_i$ and $\Sigma_i a_{i+1} m_i$ which differ by less than ϵ, and thus can be computed with any degree of accuracy. In determining the measure of a point set—and this is the more essential modification—Lebesgue covers the set with infinite sequences, rather than finite ensembles, of intervals. Thus, to the set of rational x in the interval $0 \leqq x \leqq 1$ no measure could be ascribed before Lebesgue. But these rational numbers can be arranged in a denumerable sequence $a_1, a_2, a_3, \cdots$, and, after choosing a positive number ϵ as small as one likes, one can surround the point a_n by an interval of length $\epsilon/2^n$ with the center a_n. Thus the whole set of rational points is enclosed in a

sequence of intervals of total length

$$\epsilon(1/2 + 1/2^2 + 1/2^3 + \cdots) = \epsilon;$$

and according to Lebesgue's definition its measure is therefore less than (the arbitrary positive) ϵ and hence zero. The notion of *probability* is tied to that of measure, and for this reason mathematical statisticians are deeply interested in measure theory. Lebesgue's idea has been generalized in several directions. The two fundamental operations one can perform with sets are: forming the intersection and the union of given sets, and thus sets may be considered as elements of a "*Boolean algebra*" with these two operations, the properties of which may be laid down in a number of axioms resembling the arithmetical axioms for addition and multiplication. Hence one of the questions which has occupied the more axiomatically minded among the mathematicians and statisticians is concerned with the introduction of measure in abstract Boolean algebras.

Lebesgue's integral is important in our present context, because those real-valued functions $f(x)$ of a real variable x ranging over the interval $0 \leqq x \leqq 1$, the squares of which are Lebesgue-integrable, form a complete Hilbert space —provided two functions $f(x)$, $g(x)$ are considered equal if those values x for which $f(x) \neq g(x)$ form a set of measure zero (Riesz-Fischer theorem).

The mechanical equations for a system of n degrees of freedom in Hamilton's form uniquely determine the state tP at the moment t if the state P at the moment $t = 0$ is given. Such is the precise formulation of the law of causality in mechanics. The possible states P form the points of a $(2n)$-dimensional phase space, and for a fixed t and an arbitrary P the transition $P \to tP$ is a measure-preserving mapping (t). These transformations form a group: $(t_1)(t_2) = (t_1 + t_2)$. For a given P and a variable t the point tP describes the consecutive states which this system assumes if at the moment $t = 0$ it is in the state P. Considering P as a particle of a $(2n)$-dimensional fluid which fills the phase-space and ascribing to the particle P the position tP at the time t, one obtains the picture of an incompressible fluid in stationary flow. The statistical derivation of the laws of thermodynamics makes use of the so-called *ergodic hypothesis* according to which the path of an arbitrary individual particle P (excepting initial states P which form a set of measure zero) covers the phase-space (or at least that $(2n - 1)$-dimensional sub-space of it where the energy has a given value) everywhere dense, so that in the course of its history the probability of finding it in this or that part of the space is the same for any parts of equal measure. Nineteenth century mathematics seemed to be a long way off from proving this hypothesis with any degree of generality. Strangely enough it was proved shortly after the transition from classical to quantum mechanics had rendered the hypothesis almost valueless, and it was proved by making use of the mathematical apparatus of quantum physics. Under the influence of the mapping (t), $P \to tP$, any function $f(P)$ in phase-space is transformed into the function $f' = U_t f$, defined by the equation $f'(tP) = f(P)$. The U_t form a group of

operators in the Hilbert space of arbitrary functions $f(P)$, $U_{t_1}U_{t_2} = U_{t_1+t_2}$, and application of spectral decomposition to this group enabled J. von Neumann to deduce the ergodic hypothesis with two provisos: (1) Convergence of a sequence of functions $f_n(P)$ toward a function $f(P)$, $f_n \to f$, is understood (as it would in quantum mechanics, namely) as convergence in Hilbert space where it means that the total integral of $(f_n - f)^2$ tends to zero with $n \to \infty$; (2) one assumes that there are no subspaces of the phase-space which are invariant under the group of transformations (t) except those spaces that are in Lebesgue's sense equal either to the empty or the total space. Shortly afterwards proofs were also given for other interpretations of the notion of convergence.

The laws of nature can either be formulated as differential equations or as "principles of variation" according to which certain quantities assume extremal values under given conditions. For instance, in an optically homogeneous or non-homogeneous medium the light travels along that road from a given point A to a given point B for which the time of travel assumes minimal value. In potential theory the quantity which assumes a minimum is the so-called Dirichlet integral. Attempts to establish directly the existence of a minimum had been discouraged by Weierstrass' criticism in the 19th century. Our century, however, restored the direct methods of the Calculus of Variation to a position of honor after Hilbert in 1900 gave a direct proof of the Dirichlet principle and later showed how it can be applied not only in establishing the fundamental facts about functions and integrals ("algebraic" functions and "abelian" integrals) on a compact Riemann surface (as Riemann had suggested 50 years earlier) but also for deriving the basic propositions of the *theory of uniformization*. That theory occupies a central position in the theory of functions of one complex variable, and the first decade of the 20th century witnessed the first proofs by P. Koebe and H. Poincaré of these propositions conjectured about 25 years before by Poincaré himself and by Felix Klein. As in an Euclidean vector space of finite dimensionality, so in the Hilbert space of infinitely many dimensions, this fact is true: Given a linear (complete) subspace E, any vector may be split in a uniquely determined manner into a component lying in E (orthogonal projection) and one perpendicular to E. Dirichlet's principle is nothing but a special case of this fact. But since the function-theoretic applications of orthogonal projection in Hilbert space which we alluded to are closely tied up with topology we had better turn first to a discussion of this important branch of modern mathematics: topology.

8. Topology and harmonic integrals. Essential features of the modern approach to *topology* can be brought to light in its connection with the, only recently developed, theory of *harmonic integrals*. Consider a stationary magnetic field h in a domain G which is free from electric currents. At every point of G it satisfies two differential conditions which in the usual notations of vector analysis are written in the form $\operatorname{div} h = 0$, $\operatorname{rot} h = 0$. A field of this type is called harmonic. The second of these conditions states that the line integral of

h along a closed curve (cycle) C, $\int_C h$, vanishes provided C lies in a sufficiently small neighborhood of an arbitrary point of G. This implies $\int_C h = 0$ for any cycle C in G that is the boundary of a surface in G. However, for an arbitrary cycle C in G the integral is equal to the electric current surrounded by C.

Let the phrase "C homologous to zero," $C \sim 0$, indicate that the cycle C in G bounds a surface in G. One can travel over a cycle C in the opposite sense, thus obtaining $-C$, or travel over it $2, 3, \cdots$ times, thus obtaining $2C, 3C, \cdots$; and cycles may be added and subtracted from each other (if one does not insist that cycles are of one piece). Two cycles C, C' are called homologous, $C \sim C'$, if $C - C' \sim 0$. Note that $C \sim 0$, $C' \sim 0$ imply $-C \sim 0$, $C + C' \sim 0$. Hence the cycles form a commutative group under addition, the "Betti group," if homologous cycles are considered as one and the same group element. These notions of *cycles and their homologies* may be carried over from a three-dimensional domain in Euclidean space to any n-dimensional manifold, in particular to closed (compact) manifolds like the two-dimensional surfaces of the sphere or the torus; and on an n-dimensional manifold we can speak not only of 1-dimensional, but also of 2-, 3-, $\cdots$, n-dimensional cycles. The notion of a harmonic vector field permits a similar generalization, harmonic tensor field (harmonic form) of rank r ($r = 1, 2, \cdots, n$), provided the manifold bears a Riemannian metric, an assumption the meaning of which will be discussed later in the section on geometry. Any tensor field (linear differential form) of rank r may be integrated over an r-dimensional cycle.

The fundamental problem of *homology theory* consists in determining the structure of the Betti group, not only for 1-, but also for 2-, $\cdots$, n-dimensional cycles, in particular in determining the number of linearly independent cycles (Betti number). [ν cycles $C_1, \cdots, C_\nu$ are linearly independent if there exists no homology $k_1C_1 + \cdots + k_\nu C_\nu \sim 0$ with integral coefficients k except $k_1 = \cdots = k_\nu = 0$.] The fundamental theorem for harmonic forms on compact manifolds states that, given ν linearly independent cycles $C_1, \cdots, C_\nu$, there exists a harmonic form h with pre-assigned periods

$$\int_{C_1} h = \pi_1, \cdots, \qquad \int_{C_\nu} h = \pi_\nu.$$

H. Poincaré developed the algebraic apparatus necessary to formulate exactly the notions of cycle and homology. In the course of the twentieth century it turned out that in most problems co-homologies are easier to handle than homologies. I illustrate this for 1-dimensional cycles. A line C_1 leading from a point p_1 to p_2, when followed by a line C_2 leading from p_2 to a third point p_3, gives rise to a line $C_1 + C_2$ leading from p_1 to p_3. The line integral $\int_C h$ of a given vector field h along an arbitrary (closed or open) line C is an additive function $\phi(C)$ of C, $\phi(C_1 + C_2) = \phi(C_1) + \phi(C_2)$. If moreover rot h vanishes everywhere, then $\phi(C) = 0$ for any closed line C that lies in a sufficiently small neighborhood of a point, whatever this point may be. Any real-

valued function ϕ satisfying these two conditions may be called an abstract integral. The co-homology $\phi \sim 0$ means that $\phi(C) = 0$ for any closed line C, and thus it is clear what the co-homology $k_1\phi_1 + \cdots + k_r\phi_r \sim 0$ with arbitrary real coefficients $k_1, \cdots, k_r$ means. The homology $C \sim 0$ could now be defined, not by the condition that the cycle C bounds, but by the requirement that $\phi(C) = 0$ for every abstract integral ϕ. With the convention that any two abstract integrals ϕ, ϕ' are identified if $\phi - \phi' \sim 0$, these integrals form a vector space, and the dimensionality of this vector space is now introduced as the Betti number. And the fundamental theorem for harmonic integrals on a compact manifold now asserts that for any given abstract integral ϕ there exists one and only one harmonic vector field h whose integral is cohomologous to ϕ, $\int_C h = \phi(C)$, for every cycle C (realization of the abstract integral in concreto by a harmonic integral).

J. W. Alexander discovered an important result connecting the Betti numbers of a manifold M that is embedded in the n-dimensional Euclidean space R_n with the Betti numbers of the complement $R_n - M$ (*Alexander's duality theorem*).

The difficulties of topology spring from the double aspect under which one can consider continuous manifolds. Euclid looked upon a figure as an assemblage of a finite number of geometric elements, like points, straight lines, circles, planes, spheres. But after replacing each line or surface by the set of points lying on it one may also adopt the set-theoretic view that there is only one sort of elements, points, and that any (in general infinite) set of points can serve as a figure. This modern standpoint obviously gives geometry far greater generality and freedom. In topology, however, it is not necessary to descend to the points as the ultimate atoms, but one can construct the manifold like a building from "blocks" or cells, and a finite number of such cells serving as units will do, provided the manifold is compact. Thus it is possible here to revert to a treatment in Euclid's "finitistic" style (combinatorial topology).

On the first standpoint, manifold as a point set, the task is to formulate that *continuity* by which a point p approaching a given point p_0 becomes gradually indistinguishable from p_0. This is done by associating with p_0 the *neighborhoods* of p_0, an infinite shrinking sequence of sub-sets $U_1 \supset U_2 \supset U_3 \supset \cdots$, all containing p_0. [$U \supset V$ means: the set U contains V.] For example, in a plane referred to Cartesian coordinates x, y we may choose as the nth neighborhood U_n of a point $p_0 = (x_0, y_0)$ the interior of the circle of radius $1/2^n$ around p_0. The notion of convergence, basic for all continuity considerations, is defined in terms of the sequence of neighborhoods as follows: A sequence of points $p_1, p_2, \cdots$ converges to p_0 if for every natural number n there is an N so that all points p_ν with $\nu > N$ lie in the nth neighborhood U_n of p_0. Of course, the choice of the neighborhoods U_n is arbitrary to a certain extent. For instance, one could also have chosen as the nth neighborhood V_n of (x_0, y_0) the square of side $2/n$ around (x_0, y_0), to which a point (x, y) belongs, if

$$-1/n < x - x_0 < 1/n, \qquad -1/n < y - y_0 < 1/n.$$

However the sequence V_n is equivalent to the sequence U_n in the sense that for every n there is an n' such that $U_{n'} \subset V_n$ (and thus $U_\nu \subset V_n$ for $\nu \geqq n'$), and also for every m an m' such that $V_{m'} \subset U_m$; and consequently the notion of convergence for points is the same, whether based on the one or the other sequence of neighborhoods. It is clear how to define continuity of a mapping of one manifold into another. A one-to-one mapping of two manifolds upon each other is called topological if continuous in both directions, and two manifolds that can be mapped topologically upon each other are topologically equivalent. Topology investigates such properties of manifolds as are invariant with respect to topological mappings (in particular with respect to continuous deformations).

A continuous function $y=f(x)$ may be approximated by piecewise linear functions. The corresponding device in higher dimensions, the method of *simplicial approximations* of a given continuous mapping of one manifold into another, is of great importance in set-theoretic topology. It has served to develop a general *theory of dimensions*, to prove the topological invariance of the Betti groups, to define the decisive notion of the degree of mapping ("Abbildungsgrad," L. E. J. Brouwer) and to prove a number of interesting fixed point theorems. For instance, a continuous mapping of a square into itself has necessarily a fixed point, *i.e.*, a point carried by the mapping into itself. Given two continuous mappings of a (compact) manifold M into another M', one can ask more generally for which points p on M both images on M' coincide. A famous formula by S. Lefschetz relates the "total index" of such points with the homology theory of cycles on M and M'.

Application of fixed point theorems to functional spaces of infinitely many dimensions has proved a powerful method to establish the existence of solutions for non-linear differential equations. This is particularly valuable, because the hydrodynamical and aerodynamical problems are almost all of this type.

Poincaré found that a satisfactory formulation of the homology theory of cycles was possible only from the second standpoint where the n-dimensional manifold is considered as a conglomerate of n-dimensional cells. The boundary of an n-dimensional cell (n-cell) consists of a finite number of $(n-1)$-cells, the boundary of an $(n-1)$-cell consists of a finite number of $(n-2)$-cells, *etc.* The *combinatorial skeleton* of the manifold is obtained by assigning symbols to these cells and then stating in terms of their symbols which $(i-1)$-cells belong to the boundary of any of the occurring i-cells ($i = 1, 2, \cdots, n$). From the cells one descends to the points of the manifold by a repeated process of sub-division which catches the points in an ever finer net. Since this subdivision proceeds according to a fixed combinatorial scheme, the manifold is in topological regard completely fixed by its combinatorial skeleton. And at once the question arises under what circumstances two given combinatorial skele-

tons represent the same manifold, *i.e.*, lead by iterated sub-division to topologically equivalent manifolds. We are far from being able to solve this fundamental problem. Algebraic topology, which operates with the combinatorial skeletons, is in itself a rich and beautiful theory, linked in various ways with the basic notions and theorems of algebra and group theory.

The connection between algebraic and set-theoretic topology is fraught with serious difficulties which are not yet overcome in a quite satisfactory manner. So much, however, seems clear that one had better start, not with a division into cells, but with a covering by patches which are allowed to overlap. From such a pattern the fundamental topologically invariant concepts are to be developed. The above notion of an abstract integral, which relates homology and co-homology, is an indication; it can indeed be used for a direct proof of the invariance of the first Betti number without the tool of simplicial approximation.

9. Conformal mapping, meromorphic functions, Calculus of Variation in the large. Homology theory, in combination with the Dirichlet principle or the method of orthogonal projection in Hilbert space, leads to the theory of harmonic integrals, in particular for the lowest dimension $n=2$ to the theory of abelian integrals on Riemann surfaces. But for Riemann surfaces the Dirichlet principle also yields the fundamental facts concerning uniformization of analytic functions of one variable if one combines it with the *homotopy* (not homology) *theory* of closed curves. Whereas a cycle is homologous to zero if it bounds, it is homotopic to zero if it can be contracted into a point by continuous deformation. The homotopy theory of 1- and more-dimensional cycles has recently come to the fore as an important branch of topology, and the group-theoretic aspect of homotopy has led to some surprising discoveries in abstract group theory. Homotopy of 1-dimensional cycles is closely related with the idea of the *universal covering manifold* of a given manifold. Given a continuous mapping $p \rightarrow p'$ of one manifold M into another M', the point p' may be considered as the trace or projection in M' of the arbitrary point p on M, and thus M becomes a manifold covering M'. There may be no point or several points p on M which lie over a given point p' of M' (which are mapped into p'). The mapping is without ramifications if for any point p_0 of M it is one-to-one (and continuous both ways) in a sufficiently small neighborhood of p_0. Let p_0 be a point on M, p_0' its trace on M', and C' a curve on M' beginning at p_0'. If M covers M' without ramifications we can follow this curve on M by starting at p_0, at least up to a certain point where we would run against a "boundary of M relative to M'." Of chief interest are those covering manifolds M over a given M' for which this never happens and which therefore cover M' without ramifications and relative boundaries. The best way of defining the central topological notion "simply connected" is by describing a simply connected manifold as one having no other unramified unbounded covering but itself. There is a strongest of all unramified unbounded covering manifolds, the universal

covering manifold, which can be described by the statement that on it a curve C is closed only if its trace C' is (closed and) homotopic to zero. The proof of the fundamental theorem on uniformization consists of two parts: (1) constructing the universal covering manifold of the given Riemann surface, (2) constructing by means of the Dirichlet principle a one-to-one conformal mapping of the covering manifold upon the interior of a circle of finite or infinite radius.

All we have discussed so far in our account of analysis, is in some way tied up with operators and projections in Hilbert space, the analogue in infinitely many dimensions of Euclidean space. In H. Minkowski's *Geometry of Numbers* distances $|AB|$, which are different from the Euclidean distance but satisfy the axioms that $|BA| = |AB|$ and that in a triangle ABC the inequality $|AC| \leq |AB| + |BC|$ holds, were used to great advantage for obtaining numerous results concerning the solvability of inequalities by integers. We do not find time here to report on the progress of this attractive branch of number theory during the last fifty years. In infinitely many dimensions spaces endowed with a metric of this sort, of a more general nature than the Euclid-Hilbert metric, have been introduced by Banach, not however for number-theoretic but for purely analytic purposes. Whether the importance of the subject justifies the large number of papers written on *Banach spaces* is perhaps questionable.

The Dirichlet principle is but the simplest example of the direct methods of the Calculus of Variation as they came into use with the turn of the century. It was by these methods that the theory of *minimal surfaces*, so closely related to that of analytic functions, was put on a new footing. What we know about non-linear differential equations has been obtained either by the topological fixed point method (see above) or by the so-called continuity method or by constructing their solutions as extremals of a suitable functional.

A continuous function on an n-dimensional compact manifold assumes somewhere a minimum and somewhere else a maximum value. Interpret the function as altitude. Besides summit (local maximum) and bottom (local minimum) one has the further possibility of a saddle point (pass) as a point of "stationary" altitude. In n dimensions the several possibilities are indicated by an inertial index k which is capable of the values $k = 0, 1, 2, \cdots, n$, the value $k = 0$ a minimum and $k = n$ characterizing a maximum. Marston Morse discovered the inequality $M_k \geqq B_k$ between the number M_k of stationary points of index k and the Betti number B_k of linearly independent homology classes of k-dimensional cycles. In their generalization to functional spaces these relations have opened a line of study adequately described as *Calculus of Variation in the large.*

Development of the theory of uniformization for analytic functions led to a closer investigation of *conformal mapping* of 2-dimensional manifolds in the large, which resulted in a number of theorems of surprising simplicity and beauty. In the same field there is to register an enormous extension of our

knowledge of the behavior of *meromorphic functions, i.e.*, single-valued analytic functions of the complex variable z which are regular everywhere with the exception of isolated "poles" (points of infinity). Towards the end of the previous century Riemann's zeta function had provided the stimulus for a deeper study of "entire functions" (functions without poles). The greatest stride forward, both in methods and results, was marked by a paper on meromorphic functions published in 1925 by the Finnish mathematician Rolf Nevanlinna. Besides meromorphic functions in the z-plane one can study such functions on a given Riemann surface; and in the way in which the theory of algebraic functions (equal to meromorphic functions on a compact Riemann surface) as a theory of algebraic curves in two complex dimensions may be generalized to any number of dimensions, so one can pass from meromorphic functions to meromorphic curves.

The theory of *analytic functions of several complex variables*, in spite of a number of deep results, is still in its infancy.

10. Geometry. After having dealt at some length with the problems of analysis and topology I must be brief about geometry. Of subjects mentioned before, minimal surfaces, conformal mapping, algebraic manifolds and the whole of topology could be subsumed under the title of geometry. In the domain of *elementary axiomatic geometry* one strange discovery, that of von Neumann's pointless "continuous geometries" stands out, because it is intimately interrelated with quantum mechanics, logic and the general algebraic theory of "lattices." The 1-, 2-, $\cdots$, n-dimensional linear manifolds of an n-dimensional vector space form the 0-, 1-, $\cdots$, $(n-1)$-dimensional linear manifolds in an $(n-1)$-dimensional projective point space. The usual axiomatic foundation of projective geometry uses the points as the primitive elements or atoms of which the higher-than-zero-dimensional manifolds are composed. However, there is possible a treatment where the linear manifolds of all dimensions figure as elements, and the axioms deal with the relation "B contains A" ($A \subset B$) between these elements and the operation of intersection, $A \cap B$, and of union, $A \cup B$, performed on them; the union $A \cup B$ consists of all sums $x + y$ of a vector x in A and a vector y in B. In quantum logic this relation and these operations correspond to the relation of implication ("The statement A implies B") and the operations 'and,' 'or' in classical logic. But whereas in classical logic the distributive law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

holds, this is not so in quantum logic; it must be replaced by the weaker axiom:

$$\text{If } C \subset A \text{ then } A \cap (B \cup C) = (A \cap B) \cup C.$$

On formulating the axioms without the implication of finite dimensionality one will come across several possibilities; one leads to the Hilbert space in which quantum mechanics operates, another to von Neumann's continuous geometry

with its continuous scale of dimensions, in which elements of arbitrarily low dimensions exist but none of dimension zero.

The most important development of geometry in the twentieth century took place in differential geometry and was stimulated by general relativity, which showed that the world is a 4-dimensional manifold endowed with a Riemannian metric. A piece of an n-dimensional manifold can be mapped in one-to-one continuous fashion upon a piece of the n-dimensional "arithmetical space" which consists of all n-uples $(x_1, x_2, \cdots, x_n)$ of real numbers x_i. A *Riemann metric* assigns to a line element which leads from the point $P = (x_1, \cdots, x_n)$ to the infinitely near point $P' = (x_1 + dx_1, \cdots, x_n + dx_n)$ a distance ds the square of which is a quadratic form of the relative coordinates dx_i,

$$ds^2 = \sum g_{ij}dx_i dx_j, \qquad (i, j = 1, \cdots, n)$$

with coefficients g_{ij} depending on the point P but not on the line element. This means that, in the infinitely small, Pythagoras' theorem and hence Euclidian geometry are valid, but in general not in a region of finite extension. The line elements at a point may be considered as the infinitesimal vectors of an n-dimensional vector space in P, the tangent space or the compass at P; indeed an arbitrary (differentiable) transformation of the coordinates x_i induces a linear transformation of the components dx_i of any line element at a given point P. As Levi-Civita found in 1915 the development of Riemannian geometry hinges on the fact that a Riemannian metric uniquely determines an infinitesimal parallel displacement of the vector compass at P to any infinitely near point P'. From this a general scheme for differential geometry arose in which each point P of the manifold is associated with a homogeneous space Σ_P described by a definite group of "authomorphisms," this space now taking over the role of the tangent space (whose group of authomorphisms consists of all non-singular linear transformations). One assumes that one knows how this associated space Σ_P is transferred by infinitesimal displacement to the space $\Sigma_{P'}$ associated with any infinitely near point P'. The most fundamental notion of Riemannian geometry, that of curvature, which figures so prominently in Einstein's equations of the gravitational field, can be carried over to this general scheme. Thus one has erected general differential affine, projective, conformal, geometries, *etc.* One has also tried by their structures to account for the other physical fields existing in nature beside the gravitational one, namely the electromagnetic field, the electronic wave-field and further fields corresponding to the several kinds of elementary particles. But it seems to the author that so far all such speculative attempts of building up a unified field theory have failed. There are very good reasons for interpreting gravitation in terms of the basic concepts of differential geometry. But it is probably unsound to try to "geometrize" all physical entities.

Differential geometry in the large is an interesting field of investigation which relates the differential properties of a manifold with its topological structure. The schema of differential geometry explained above with its associated spaces

Σ_P and their displacements has a purely topological kernel which has recently developed under the name of *fibre spaces* into an important topological technique.

Our account of progress made during the last fifty years in analysis, geometry and topology had to touch on many special subjects. It would have failed completely had it not imparted to the reader some feeling of the close relationship connecting all these mathematical endeavors. As the last example of fibre spaces (beside many others) shows, this unity in diversity even makes a clear-cut division into analysis, geometry, topology (and algebra) practically impossible.

11. Foundations. Finally a few words about the *foundations of mathematics.* The nineteenth century had witnessed the critical analysis of all mathematical notions including that of natural numbers to the point where they got reduced to pure logic and the ideas "set" and "mapping." At the end of the century it became clear that the unrestricted formation of sets, sub-sets of sets, sets of sets *etc.*, together with an unimpeded application to them as to the original elements of the logical quantifiers "there exists" and "all" [cf. the sentences: the (natural) number n is even if there exists a number x such that $n = 2x$; it is odd if n is different from $2x$ for all x] inexorably leads to antinomies. The three most characteristic contributions of the twentieth century to the solution of this Gordian knot are connected with the names of L. E. J. Brouwer, David Hilbert and Kurt Gödel. Brouwer's critique of "mathematical existentialism" not only dissolved the antinomies completely but also destroyed a good part of classical mathematics that had heretofore been universally accepted.

If only the historical event that somebody has succeeded in constructing a (natural) number n with the given property P can give a right to the assertion that "there exists a number with that property" then the alternative that there either exists such a number or that all numbers have the opposite property non-P is without foundation. The principle of excluded middle for such sentences may be valid for God who surveys the infinite sequence of all natural numbers, as it were, with one glance, but not for human logic. Since the quantifiers "there is" and "all" are piled upon each other in the most manifold way in the formation of mathematical propositions, Brouwer's critique makes almost all of them meaningless, and therefore Brouwer set out to build up a new mathematics which makes no use of that logical principle. I think that everybody has to accept Brouwer's critique who wants to hold on to the belief that mathematical propositions tell the sheer truth, truth based on evidence. At least Brouwers' opponent, Hilbert, accepted it tacitly. He tried to save classical mathematics by converting it from a system of meaningful propositions into a game of meaningless formulas, and by showing that this game never leads to two formulas, F and non-F, which are inconsistent. Consistency, not truth, is his aim. His attempts at proving consistency revealed the astonishingly complex logical structure of mathematics. The first steps were promising indeed. But

then Gödel's discovery cast a deep shadow over Hilbert's enterprise. Consistency itself may be expressed by a formula. What Gödel showed was this: If the game of mathematics is actually consistent then the formula of consistency cannot be proved within this game. How can we then hope to prove it at all?

This is where we stand now. It is pretty clear that our theory of the physical world is not a description of the phenomena as we perceive them, but is a bold symbolic construction. However, one may be surprised to learn that even mathematics shares this character. The success of the anti-phenomenological constructive method is undeniable. And yet the ultimate foundations on which it rests remain a mystery, even in mathematics.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, *New York University*

Material for this department should be sent to F. A. Ficken, Institute for Mathematics and Mechanics, New York University, 45 Fourth Ave., New York 3, New York.

ON EVEN NUMBERS m DIVIDING 2^m-2

N. G. W. H. BEEGER, Amsterdam, Netherlands

In a recent paper P. Erdös [1] notes that D. H. Lehmer has recently detected the first case of an even number $m=2.73.1103=161038$ which satisfies the congruence

$$2^m - 2 \equiv 0 \pmod{m}. \tag{1}$$

It is the purpose of this paper to give three more such numbers and to prove that there exists an infinity.

In the first place it is clear from (1) that m is not divisible by 4 so that setting $m=2n$ (n odd), congruence (1) is equivalent to

$$2^{2n-1} - 1 \equiv 0 \pmod{n}. \tag{2}$$

In what follows we denote by e_p the exponent of 2 (mod p), p a prime, that is, the least positive x for which $2^x \equiv 1 \pmod{p}$. It is well known that e_p divides $p-1$ as well as any other x for which $2^x \equiv 1 \pmod{p}$. In particular if p is any prime factor of n in (2) then clearly e_p is odd since it divides $2n-1$.

THEOREM 1. *If* (2) *holds then n has at least two prime factors.*

Proof. If possible let $n=p^k$, then

$$2p^k - 1 = 2(p^k - 1) + 1$$

is divisible by e_p. But e_p divides $p-1$ and hence p^k-1. Therefore $e_p=1$, which is impossible.

THEOREM 2. *If* (2) *holds and if* n *is divisible by the square of a prime* p, *then either* $p=3511$, or $p>25000$.

Proof. Let $n=kp^2$ and let

$$2^{e_p} \equiv 1 + ph \pmod{p^2}.$$

Then by (2) e_p divides $2n-1$ so that $2n-1=ye_p$. Hence

$$1 \equiv 2^{2n-1} \equiv (2^{e_p})^y \equiv (1 + hp)^y \equiv 1 + hyp \pmod{p^2}.$$

Thus p divides hy. But p does not divide $ye_p=2n-1=2kp^2-1$ so that p does not divide y. Hence p divides h or in other words

$$2^{e_p} \equiv 1 \pmod{p^2}.$$

The only primes less than 25000 having this property are 1093 and 3511 with exponents 364 and 1755 respectively [2]. Since e_p must be odd the theorem follows.

THEOREM 3. If $n=p_1p_2 \cdots p_k$, *then for* (2) *to hold it is necessary and sufficient that* e_{p_i} *divide* $2np_i^{-1}-1$ *for* $i=1, 2, \cdots, k$.

Proof. The congruence (2) is equivalent to the condition that $2n-1$ is divisible by each e_{p_i}. But

$$2n - 1 = 2np_i^{-1}(p_i - 1) + 2np_i^{-1} - 1.$$

Three new solutions of (1) may be obtained by applying Theorem 3 to the case $n=23.31.p$. In this case

$$e_{23} = 11, \qquad e_{31} = 15,$$

so that by Theorem 3, p must satisfy

$$62p \equiv 1 \pmod{11}$$
$$46p \equiv 1 \pmod{15}$$
$$1426p \equiv 1 \pmod{e_p}.$$

The first two conditions imply that

$$p \equiv 151 \pmod{330} \tag{3}$$

while the last implies that e_p is some divisor k of 1425, whose divisors are

$$k = 1, 3, 5, 15, 19, 25, 57, 75, 95, 285, 475, 1425.$$

Tables of the factors of 2^k-1 give the primes having specified exponents, the primitive prime factors of 2^k-1 being the only primes having exponent k. Because the factorization of 2^k-1 is incomplete for $n=95,285,475$ and 1425 not all primes p for which e_p divides 1425 are known. Most of those which are known fail to satisfy (3). In fact only three values $p=151$, 1801, 100801 satisfy (3). [3] This gives us the three values of m satisfying (1)

215326, 2568226, 143742226.

Finally we have

THEOREM 4. *Congruence* (1) *has an infinity of solutions.*

Proof. The proof consists in showing that given any solution m of (1) there exists a prime p depending on m such that mp is also a solution.

In fact let $m=2n$ be any solution of (1) so that (2) holds and let p be any primitive prime factor [4] of $2^{2n-1}-1$ so that $e_p=2n-1$ and

$$2^{2n-1} \equiv 1 \pmod{p}.$$

Since e_p divides $p-1$

$$p = (2n-1)x + 1 > n.$$

Hence p and n are coprime. To show that mp is a solution of (1) it therefore suffices to establish that

$$2^{2pn-1} \equiv 1 \pmod{m}$$
$$2^{2pn-1} \equiv 1 \pmod{p}$$

or to show that $2pn-1$ is divisible by $2n-1$ and e_p respectively. But $e_p=2n-1$ and

$$2pn - 1 = 2n - 1 + 2n(p-1) = (2n-1)(1+2nx).$$

This completes the proof of Theorem 4.

References

1. This MONTHLY v. 57, p. 405 (1950).
2. Messenger of Math. v. 51, pp. 149–152 (1922); v. 55, pp. 17–24 (1925). Nieuw Arch. v. Wskde. v. 20, pp. 51–54 (1939). Unpublished result of D. H. Lehmer.
3. *Factorization of* $y^n \pm 1$, A. J. C. Cunningham and H. J. Woodall, London, 1925.
4. That p exists follows from a theorem of Bang, Tidskrift of Mat. s. 5, v. 4, 1886, 130–137.

AN ELEMENTARY METHOD FOR EVALUATING AN INFINITE INTEGRAL

M. R. SPIEGEL, Rensselaer Polytechnic Institute

1. Introduction. In the March, 1951, issue of this MONTHLY W. Kozakiewicz proved the relation

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} \tag{1}$$

by appealing to the Riemann theorem; namely, that if $f(x)$ is Riemann integrable in $a \leqq x \leqq b$ then

$$\lim_{k\to\infty} \int_a^b f(x) \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} dx = 0. \tag{2}$$

The purpose of this article is two-fold. The first is to point out how one may avoid the Riemann theorem in Kozakiewicz' paper by use of a more elementary theorem. The second is to arrive at the relation (1) by evaluating directly an equivalent integral, namely,

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx, \tag{3}$$

by making use of our theorem.

2. Theorem. The more elementary theorem may be stated as follows:

THEOREM. *If $f(x)$ is Riemann integrable and bounded in $a \leqq x \leqq b$, then*

$$\lim_{k\to\infty} \int_a^b f(x) \left\{\frac{1-\cos kx}{k}\right\} dx = 0. \tag{4}$$

Proof.

$$\left| \int_a^b f(x) \left\{\frac{1-\cos kx}{k}\right\} dx \right| \leqq \frac{2(b-a)M}{k} \tag{5}$$

where M is an upper bound of $|f(x)|$ in $a \leqq x \leqq b$. Letting $k \to \infty$ in (5) we have the result (4).

3. Remark Concerning Kozakiewicz' Paper. It is possible to avoid the Riemann theorem in Kozakiewicz' paper, since use is made of the Riemann theorem there only to prove that

$$\lim_{k\to\infty} \int_0^\pi \left[\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right] \sin kx\,dx = 0 \tag{6}$$

where $k = n + 1/2$. This may be proved directly by use of our theorem. For we may show that

$$f(x) = \frac{d}{dx}\left[\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right]$$

satisfies the conditions of our theorem by reasoning analogous to that used immediately following (10). Thus

$$\lim_{k\to\infty} \int_0^\pi \frac{d}{dx}\left[\frac{1}{x} - \frac{1}{2\sin\frac{x}{2}}\right]\left[\frac{1-\cos kx}{k}\right] dx = 0.$$

An integration by parts easily yields the result (6).

4. Evaluation of the Integral. We shall now prove (1) by evaluating directly the equivalent integral (3), (the equivalence may be proved easily by an integration by parts). We make use of certain easily proved trigonometric formulas; namely, if

$$\sigma_n(x) = \tfrac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx \tag{7}$$

then

$$S_k(x) = \frac{1}{k}\left[\sigma_0(x) + \sigma_1(x) + \cdots + \sigma_{k-1}(x)\right] = \frac{1}{2k}\left[\frac{\sin \frac{kx}{2}}{\sin \frac{x}{2}}\right]^2 \tag{8}$$

and

$$\int_0^{\pi} S_k(x)dx = \frac{\pi}{2}. \tag{9}$$

From (9) we have

$$\int_0^{\pi} \frac{1}{k}\left\{\frac{1 - \cos kx}{4 \sin^2 \frac{x}{2}}\right\} dx = \frac{\pi}{2}. \tag{10}$$

Calling

$$f(x) = \frac{1}{x^2} - \frac{1}{4 \sin^2 \frac{x}{2}}$$

we may show that

$$\lim_{x \to 0} f(x) = -\frac{1}{12}.$$

Hence, if we define $f(0) = -1/12$, then $f(x)$ is continuous in $0 \leqq x \leqq \pi$.

Hence, $f(x)$ satisfies the conditions of our theorem and, therefore,

$$\lim_{k \to \infty} \int_0^{\pi} \left[\frac{1}{x^2} - \frac{1}{4 \sin^2 \frac{x}{2}}\right]\left[\frac{1 - \cos kx}{k}\right] dx = 0. \tag{11}$$

Thus, by (10)

$$\lim_{k \to \infty} \int_0^{\pi} \frac{1}{x^2}\left[\frac{1 - \cos kx}{k}\right] dx = \frac{\pi}{2}. \tag{12}$$

or

$$\lim_{k\to\infty}\int_0^\tau \frac{\sin^2\frac{kx}{2}}{kx^2}\,dx=\frac{\pi}{4}.$$

This becomes, when $kx/2$ is replaced by x,

$$\lim_{k\to\infty}\int_0^{k\tau/2}\frac{\sin^2 x}{x^2}\,dx=\int_0^\infty\frac{\sin^2 x}{x^2}\,dx=\frac{\pi}{2}.$$

Since

$$\int_0^\infty\frac{\sin^2 x}{x^2}\,dx=\int_0^\infty\frac{\sin x}{x}\,dx,$$

the relation (1) follows.

A NOTE ON INEQUALITIES FOR THE NORM OF A MATRIX

P. Stein, University of Natal, Durban, South Africa

Let $A=\|a_{ij}\|$ be a square matrix of order n with real elements and $B=\frac{1}{2}(A+A')$ the associated symmetric matrix with elements $b_{ij}=\frac{1}{2}(a_{ij}+a_{ji})$. It is the purpose of this note to relate the minimum and maximum roots of B to the trace and norm of A^{-1}.

Let x be a matrix of one column and x' its transpose (a row matrix). Then $x'Ax=x'A'x$ is a scalar and

$$x'Bx=x'Ax. \tag{1}$$

If m and M are the (necessarily real) minimum and maximum characteristic roots of B and the components x_i are real, a well known theorem* gives

$$m(x'x)\leqq x'Bx\leqq M(x'x). \tag{2}$$

Applying this result successively to each of the column vectors of a square matrix X and summing the resulting inequalities yields

$$m(\text{trace } X'X)\leqq \text{trace } X'BX\leqq M(\text{trace } X'X). \tag{3}$$

In particular, for $X=(\text{adj } A)'$, $X'BX=\frac{1}{2}\Delta[(\text{adj } A)+(\text{adj } A)']$, so that

$$m\cdot N^2(\text{adj } A)\leqq\Delta\cdot\text{trace (adj } A)\leqq M\cdot N^2(\text{adj } A), \tag{4}$$

where $\Delta=$determinant of A and $N^2(A)=\text{trace } A'A=\sum_{i,j}(a_{ij})^2$ is the square of the norm of A. Using $X=A^{-1}$ if $\Delta\neq 0$,

$$m\cdot N^2(A^{-1})\leqq\text{trace } A^{-1}\leqq M\cdot N^2(A^{-1}). \tag{5}$$

In case A is symmetric, $B=A$ and m and M are the minimum and maximum

* R. Courant and D. Hilbert, Methoden der Mathematischen Physik I, Berlin 1931.

roots of A itself. An orthogonal T exists such that $TAT^{-1}=TAT'=A^*$ is diagonal and of course has the same roots, say λ_i $(i=1, \cdots, n)$, that A does. Also $N^2((A^*)^{-1})=N^2(A^{-1})$ and trace $(A^*)^{-1}=$ trace A^{-1}. Hence, (5) reduces to the obvious inequality

$$(\min \lambda_i) \sum_i (\lambda_i)^{-2} \leqq \sum_i (\lambda_i)^{-1} \leqq (\max \lambda_i) \sum_i (\lambda_i)^{-2}.$$

A further case one might mention is the case $A=I+S$, with $S'=-S$. A is then known to be non-singular. Since $B=I$, we have $m=M=1$, and the inequality (5) reduces to the equality

$$N^2(A^{-1}) = \text{trace } (A^{-1}).$$

Further, since for any square matrix M, $N^2(M)=$trace MM' we have the result that, if $S'=-S$,

$$\text{trace } (I - S^2)^{-1} = \text{trace } (I - S)^{-1}.$$

If A contains complex elements, inequalities similar to (4) and (5) may be obtained by taking $B=\frac{1}{2}(A+\overline{A}')$. B is then Hermitian. Corresponding to (3) we have the inequalities

$$m \cdot \text{trace } (\overline{X}'X) \leqq \text{trace } (\overline{X}'BX) \leqq M \cdot \text{trace } (\overline{X}'X) \tag{6}$$

and corresponding to (4), the inequality

$$m \cdot N^2(\text{adj } A) \leqq \tfrac{1}{2}\Delta \text{ trace } (\text{adj } \overline{A}) + \tfrac{1}{2}\overline{\Delta} \text{ trace } (\text{adj } A). \tag{7}$$

If $\Delta \neq 0$, corresponding to (5) we then have the inequality

$$m \cdot N^2(A^{-1}) \leqq \text{Real part } \{\text{trace } (A^{-1})\} \leqq M \cdot N^2(A^{-1}).$$

Finally, I wish to acknowledge with thanks the assistance of the referee in putting this note into a form suitable for publication.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, University of Washington

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, 39, Mass.

TAYLOR'S THEOREM IN A FIRST COURSE

C. P. NICHOLAS, U. S. Military Academy

A sophomore's first encounter with Taylor's Theorem is likely to be frustrating. If his text is one of several in common use throughout the United States, it will prove Taylor's Theorem by appealing to Rolle's Theorem, introducing for

that purpose an elaborate auxiliary function seemingly drawn out of thin air. This is a bewildering approach for beginners: it shatters their faith in the directness of mathematics.

A well-known alternative is to derive the theorem by means of n successive integrations, starting arbitrarily with the integral of $f^{(n)}(x)dx$. This also is artificial diet for an undergraduate. What he needs instead is a discussion starting along paths his own logic might suggest.

A derivation designed to appeal intuitively to sophomores is offered below. It presupposes that the student is prepared to evaluate the successive integral

$$\int_a^x \int_a^x \cdots \int_a^x (dx)^n = \frac{(x-a)^n}{n!}$$

when he meets it in due course. If not, he will need a brief preliminary explanation, but otherwise no special preparation is required.

We assume that the student has been led into the subject carefully. Thus, he understands that his problem is to find an approximate value for $f(x)$ at a point x, based on known values of $f(x)$ and its first n derivatives at a neighboring point $x=a$. Also, he knows that $f(x)$ and its first n derivatives are continuous throughout the closed interval (a, x). The unknown quantity whose approximation he seeks is represented by the ordinate at x in Figure 1.

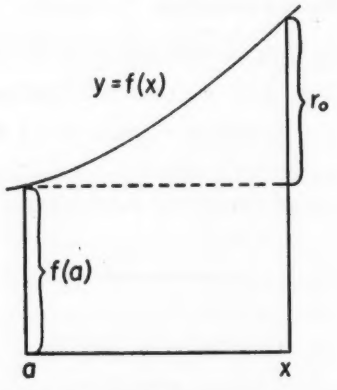

FIG. 1

Since $f(a)$ is known, clearly the heart of the problem is to find an approximation for the increment r_0. Therefore, we introduce a tool designed especially to express the value of such an increment. It is based on an elementary relation between successive derivative curves.

For each curve considered, we shall let r denote the increment in the function as x changes from a to x. A subscript to r will denote that the increment pertains to a derivative curve of order corresponding to the subscript. Thus, r_n

will denote a change in $f^{(n)}(x)$ as x changes from a to x.

Consider now the relationship between any function $G(x)$ and its first derivative, as shown between Figures 2 and 3.

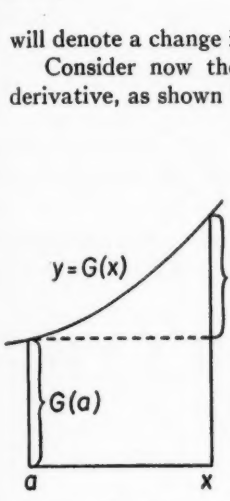

FIG. 2

$$r_0 = G(x) - G(a) \text{ (Fig. 2)}$$
$$= \int_a^x G'(x)dx$$
$$= \text{area under } G'(x)$$
$$= \text{area } \underline{A} + \text{area } \underline{B} \text{ (Fig. 3).}$$

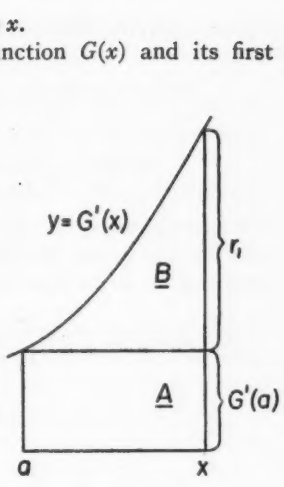

FIG. 3

Therefore:

$$r_0 = G'(a)\int_a^x dx + \int_a^x r_1 dx.$$

The foregoing result is one case of a general relationship between *any* two consecutive derivative curves. We may express it by substituting $f^{(n-1)}(x)$ for $G(x)$, thus:

(I) $$r_{n-1} = f^{(n)}(a)\int_a^x dx + \int_a^x r_n dx.$$

At one point in the ensuing discussion, we shall have occasion to use a different (and more obvious) form of the above relationship. Since $f^{(n)}(a)+r_n=f^{(n)}(x)$, we may write:

(II) $$r_{n-1} = \int_a^x f^{(n)}(x)dx.$$

We shall now use (I) and (II) to expand a function, $f(x)$, in the vicinity of $x=a$. By our previous definition of r we may state:

$$f(x) = f(a) + r_0.$$

Applying (I) to r_0, we have:

$$f(x) = f(a) + f'(a)\int_a^x dx + \int_a^x r_1 dx.$$

Again, applying (I) to r_1 we have:

$$f(x) = f(a) + f'(a)\int_a^x dx + \int_a^x \left[f''(a)\int_a^x dx + \int_a^x r_2 dx\right] dx$$

$$= f(a) + f'(a)\int_a^x dx + f''(a)\int_a^x \int_a^x (dx)^2 + \int_a^x \int_a^x r_2 (dx)^2.$$

Continuing, we apply (I) successively to r_2, r_3, r_4, and so on, until we have expanded $f(x)$ to any desired number of terms. We then stop the expansion by applying II to the final term. Thus:

$$f(x) = f(a) + f'(a)\int_a^x dx + f''(a)\int_a^x \int_a^x (dx)^2 + f'''(a)\int_a^x \int_a^x \int_a^x (dx)^3$$

$$+ \cdots + f^{n-1}(a)\int_a^x \int_a^x \cdots \int_a^x (dx)^{n-1}$$

$$+ \int_a^x \int_a^x \int_a^x \cdots \int_a^x f^{(n)}(x)(dx)^n.$$

Evaluating each integral except the last, we obtain Taylor's Formula:

$$f(x) = f(a) + f'(a)\cdot(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots$$

$$+ f^{(n-1)}(a)\frac{(x-a)^{n-1}}{(n-1)!} + R_n$$

where R_n, the remainder after n terms, is given by

$$R_n = \int_a^x \int_a^x \cdots \int_a^x f^{(n)}(x)(dx)^n.$$

Since we cannot evaluate the above integral conveniently, this form of R_n is not useful. Therefore we shall replace it by an equivalent expression more easily evaluated. We do this by substituting for the variable $f^{(n)}(x)$ a constant selected in such a way that the value of R_n will not be altered by the substitution. It can be shown* that $f^n(\xi)$ is such a constant, where ξ is some fixed but unknown value between a and x. Making this substitution, we have:

$$R_n = f^{(n)}(\xi)\int_a^x \int_a^x \cdots \int_a^x (dx)^n, \qquad a < \xi < x.$$

Integrating, we obtain the Lagrange form of the remainder:

$$R_n = f^{(n)}(\xi)\frac{(x-a)^n}{n!}, \qquad a < \xi < x.$$

* The proof is substantially the same as for the Mean Value Theorem for Integrals.

AN APPLICATION OF A FAMOUS INEQUALITY

N. S. MENDELSOHN, University of Manitoba

The proofs usually given in elementary textbooks of the fact that the sequence $S_n=(1+1/n)^n$ approaches a limit as $n\to\infty$ are either incorrect or quite messy. The correct proofs are usually based on the following facts:

(1) The sequence $S_n=(1+1/n)^n$ is an increasing function of n for positive integral n.

(2) The sequence $T_n=(1+1/n)^{n+1}$ is a decreasing function of n for positive integral n.

(3) $T_n>S_n$ for every positive integer n.

The proof of (3) is quite trivial. In this note we address ourselves to the problem of obtaining interesting proofs of (1) and (2). The proofs are based on the famous inequality which states that the arithmetic mean of a set of k positive numbers is greater than their geometric mean.

Proof of (1). Consider the set of $(n+1)$ numbers

$$1,\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ 1+\frac{1}{n},\ \cdots,\ 1+\frac{1}{n}.$$

These have an arithmetic mean of $1+1/(n+1)$ and a geometric mean of $(1+1/n)^{n/(n+1)}$. Hence

$$1+\frac{1}{n+1}>\left(1+\frac{1}{n}\right)^{n/(n+1)} \quad \text{or} \quad \left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^{n}.$$

Hence S_n increases with n.

Proof of (2). Consider the set of $(n+2)$ numbers.

$$1,\ \frac{n}{n+1},\ \frac{n}{n+1},\ \frac{n}{n+1},\ \cdots,\ \frac{n}{n+1}.$$

These have an arithmetic mean of $(n+1)/(n+2)$ and a geometric mean of $(n/n+1)^{(n+1)/(n+2)}$. Hence

$$\frac{n+1}{n+2}>\left(\frac{n}{n+1}\right)^{(n+1)/(n+2)}.$$

On taking reciprocals this becomes

$$1+\frac{1}{n+1}<\left(1+\frac{1}{n}\right)^{(n+1)/(n+2)} \quad \text{or} \quad \left(1+\frac{1}{n+1}\right)^{n+2}<\left(1+\frac{1}{n}\right)^{n+1}.$$

Hence T_n is a decreasing function of n.

The proof of (1) was first given by a first year student at the University of Toronto about 10 years ago. The proof of (2) rests with the author of this note.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 980. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a sphere through the vertices A, B, C of a tetrahedron $ABCD$ cuts the edges DA, DB, DC in A', B', C', then the distances of its center from those of the spheres $ABCD$ and $A'B'C'D$ are equal, respectively, to the radii of the spheres $A'B'C'D$ and $ABCD$.

E 981. *Proposed by David Mandelbaum, Hillside, N. J.*

Derive a formula for the sum of the first n terms of a progression in which the first term is a, each even placed term is obtained from its preceding term by multiplying by the constant u, and each odd placed term (after the first) is obtained from its preceding term by multiplying by the constant v.

E 982. *Proposed by C. W. Trigg, Los Angeles City College*

Show that there are two different sets of three congruent right cylindrical surfaces which may be folded and assembled into a cube with no open edges.

E 983. *Proposed by A. W. Goodman, University of Kentucky*

Let M and N be two points one unit apart. With M and N as centers and with unit radii draw arcs ANB and AMB. Let Q be any point on arc AMB and P_1 and P_2 any points on arc ANB such that N is the midpoint of arc P_1P_2. Show that

$$QP_1 + QP_2 \leqq 2 \leqq (QP_1)^2 + (QP_2)^2.$$

E 984. *Proposed by Joseph Rosenbaum, Hartford, Conn.*

(a) Find $f(x)$ when $f[f(x)] = x^2 - 2$.

(b) More generally, find $f_1(x)$ when $f_n(x) = x^2 - 2$, where $f_n(x)$ is defined by the relation $f_{r+1}(x) = f_1[f_r(x)]$.

E 985. *Proposed by Leo Moser, Texas Technological College*

What is the smallest integer N such that if the integers $1, 2, 3, \cdots, N$ are distributed into three classes in any manner whatsoever at least one of the classes will contain a solution of $a + b = c$.

SOLUTIONS

Inequality of K. O. Friedrichs

E 949 [1951, 37]. *Proposed by Robert Oeder, Oregon State College*

Show that if D_n is a determinant whose elements are a_{ij}, $i, j = 1, 2, \cdots, n$, and, for all i,

$$a_{ii} \geqq (1/2)\sum_{k=1}^{n} |a_{ik}|,$$

then

$$D_n \geqq a_{11}\prod_{i=2}^{n}\left(a_{ii} - \sum_{k=1}^{i-1}|a_{ik}|\right).$$

Solution by the Proposer. Olga Taussky, in *A recurring theorem on determinants*, this MONTHLY, 1949, pp. 672–675, has shown that $D_n \geqq 0$. Now

$$D_n = \begin{vmatrix} a_{11} & a_{12} & a_{13}\cdots a_{1n} \\ 0 & a_{22} - |a_{21}| & a_{23}\cdots a_{2n} \\ a_{31} & a_{32} & a_{33}\cdots a_{3n} \\ \cdot & \cdot & \cdot\ \cdots\ \cdot \\ a_{n1} & a_{n2} & a_{n3}\cdots a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13}\cdots a_{1n} \\ a_{21} & |a_{21}| & 0\ \cdots\ 0 \\ a_{31} & a_{32} & a_{33}\cdots a_{3n} \\ \cdot & \cdot & \cdot\ \cdots\ \cdot \\ a_{n1} & a_{n2} & a_{n3}\cdots a_{nn} \end{vmatrix}.$$

By Taussky's result, the second determinant in the sum is non-negative, and D_n is therefore not less than the first determinant in the sum. By applying this process to the remaining $n-2$ rows one obtains

$$D_n \geqq \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} - |a_{21}| & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} - |a_{31}| - |a_{32}| & \cdots & a_{3n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_{nn} - \sum_{k=1}^{n-1}|a_{nk}| \end{vmatrix}$$

$$= a_{11}\prod_{i=2}^{n}\left(a_{ii} - \sum_{k=1}^{i-1}|a_{ik}|\right).$$

Also solved by E. V. Haynsworth and O. E. Stanaitis.

Attention should be called to G. B. Price, *Bounds for determinants with dominant principal diagonal*, Proc. Am. Math. Soc., 1951, pp. 497–502. The result of the problem is a special case of Theorem 1 of this paper.

Powers of a Binomial Surd

E 950 [1951, 108]. *Proposed by W. R. Ransom, Tufts College*

Show that every positive integral power of $\sqrt{2}-1$ is of the form $\sqrt{m}-\sqrt{m-1}$.

Solution by S. T. Thompson, Tacoma, Washington. We shall in (1), obtain a considerably more general result. Let a and b, $a \geqq b$, be two non-negative real numbers, and n a positive integer. Set

$$p = [(a+b)^n + (a-b)^n]/2.$$

Then

$$p^2 - (a^2-b^2)^n = [(a+b)^n - (a-b)^n]^2/4.$$

It follows that

$$(a-b)^n = \sqrt{p^2} - \sqrt{p^2-(a^2-b^2)^n}. \tag{1}$$

The following special cases, of which (*C*) is the required result, are worthy of note:

(A) Take $a=\sqrt{s}$, $b=\sqrt{s-k}$, s and k positive integers, $s \geqq k$. Then p^2 is a positive integer, and

$$(\sqrt{s}-\sqrt{s-k})^n = \sqrt{p^2} - \sqrt{p^2-k^n}.$$

(B) In (A) take $k=1$. Then

$$(\sqrt{s}-\sqrt{s-1})^n = \sqrt{p^2} - \sqrt{p^2-1}.$$

(C) In (B) take $s=2$. Then

$$(\sqrt{2}-1)^n = \sqrt{p^2} - \sqrt{p^2-1}.$$

Here $p=[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n]/2$.

Also solved by A. N. Aheart, H. L. Alder, F. Bagemihl, E. W. Banhagel, A. W. Boldyreff, D. H. Browne, Aaron Buchman, I. A. Dodes, Arnold Dresden, D. G. Duncan, Ragnar Dybvik, R. E. Edwards, R. E. Ekstrom, A. L. Epstein, Leopold Flatto, Walter Fleming, Calvin Foreman, A. E. Foster, A. E. Franz, Albert Furman, Bernard Greenspan, N. G. Gunderson, Frank Harary, Vern Hoggatt, L. N. Howard, R. Huck, Ray Jurgensen, C. W. Karns, Abraham Karrass, R. S. Kingsbury, G. J. Kleinhesselink, W. J. Klimczak, Sam Kravitz, Sidney Kravitz, H. R. Leifer, A. J. Lohwater, Fred Marer, D. R. Morrison, Leo Moser, Prasert Na Nagara, C. S. Ogilvy, J. H. Oppenheim, F. D. Parker, R. G. Paxman, C. G. Phipps, C. F. Pinzka, Daniel Resch, L. A. Ringenberg, Cal Rogers, Alex Rosenberg, C. M. Sandwick, Sr., J. W. Sawyer, W. Seidel, O. D. Smith, Robert Spira, O. E. Stanaitis, Elijah Swift, G. W. Walker, Alan Wayne, J. E. Weidlich, Albert Wilansky, L. M. Winer, E. M. Zaustinsky, and the proposer.

Concerning Symmedians

E 951 [1951, 108]. *Proposed by C. S. Venkataraman, Trichur, South India*

Two circles are drawn touching the sides AB, AC of a triangle ABC at the ends of the base BC and also passing through the midpoint D of BC. If E is the other point of intersection of the circles show that AE is a symmedian of triangle ABC.

Solution by Miriam Huggins, Woman's College of the University of North Carolina. Since $\sphericalangle ACD = \sphericalangle DEC$ and $\sphericalangle ABD = \sphericalangle BED$ it follows that $\sphericalangle BEC$ and $\sphericalangle BAC$ are supplementary and therefore that $ABEC$ is cyclic. Hence $\sphericalangle BCE = \sphericalangle BAE$. But also $\sphericalangle ABE = \sphericalangle EDC$, since each is equal to $\sphericalangle BED + \sphericalangle DBE$. Therefore triangles ABE and CDE are similar, whence $BE/DE = AB/DC = AB/BD$. It now follows that triangles BED and ABD are similar. Then $\sphericalangle BAD = \sphericalangle EBD = \sphericalangle EAC$, or AE is a symmedian.

Also solved by N. Balasubramanian, Vern Hoggatt, Prasert Na Nagara, A. Sisk, O. D. Smith, and Roscoe Woods.

P-slopes

E 952 [1951, 108]. *Proposed by Albert Wilansky, Lehigh University*

Let $P:(r, \theta)$ be given in plane polar coordinates. Define the *p-slope* of the segment P_1P_2 to be $(r_2 - r_1)/(\text{distance from } P_1 \text{ to } P_2)$. If A, B, C are three collinear points, with B between A and C, show that p-slope $CB \leqq p$-slope $CA \leqq p$-slope BA. (*Note*: The inequality signs in the original statement of the problem should be reversed, as above.)

Solution by the Proposer. Let Q be the foot of the perpendicular from the origin O on the line of collinearity of A, B, C, and let P be any point on the line. Designate the lengths of OQ, OP, QP by c, r, ρ, where c and r are taken as positive but ρ is a signed length. Then $r = (\rho^2 + c^2)^{1/2}$ is a convex function of ρ. Designate the (r, ρ) coordinates of A, B, C by (r_A, ρ_A), (r_B, ρ_B), (r_C, ρ_C). By convexity

$$\frac{r_B - r_A}{|\rho_B - \rho_A|} \leqq \frac{r_C - r_A}{|\rho_C - \rho_A|} \leqq \frac{r_C - r_B}{|\rho_C - \rho_B|},$$

and the theorem is proved.

Also solved by Vern Hoggatt, F. D. Parker, and C. M. Sandwick, Sr.

Tetrahedron and Four Planes

E 953 [1951, 108]. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Four planes, drawn through a point, parallel to the faces of a tetrahedron, cut the edges of the tetrahedron in 12 points lying on a quadric surface.

Solution by S. T. Thompson, Tacoma, Washington. The four planes cut the three edges of each face in six points which, by the converse of Carnot's theorem,

are easily shown to lie on a conic. These four conics intersect in pairs in two points. Since, in general, nine points determine a quadric surface, it now follows that the four conics lie on a common quadric surface.

The analogous theorem in the plane is well known.

The theorem may be slightly generalized to the following: If a plane cuts the four faces of a tetrahedron in four lines, then any four concurrent planes passing through these four lines intersect the edges of the tetrahedron in 12 points lying on a quadric surface.

Triangular Squares

E 954 [1951, 108]. *Proposed by G. W. Walker, Buffalo, N. Y.*

Let T_c be the cth positive integer which is both a triangular number and a square number. Find T_c as a function of c.

Solution by W. J. Klimczak, Iola Sanatorium, Rochester, N. Y. The kth triangular number and the nth square are equal when $k(k+1)/2=n^2$. Solving this as a quadratic in k we find $k=[-1\pm\sqrt{1+8n^2}]/2$. Since k and n are positive integers we must have $1+8n^2=x^2$, say, where x is a positive integer. From the solution $n=1$, $x=3$ of this equation we obtain as the general solution for n,

$$n = [(3+\sqrt{8})^c - (3-\sqrt{8})^c]/2\sqrt{8}.$$

Then

$$T_c = n^2 = [(17+12\sqrt{2})^c + (17-12\sqrt{2})^c - 2]/32.$$

The first five values of T_c are 1, 36, 1225, 41616, and 1413721.

Also solved by J. H. Braun, D. H. Browne, Sam Kravitz, Leo Moser, Prasert Na Nagara, P. A. Piza, L. A. Ringenberg, J. E. Sanders, C. M. Sandwick, Sr., E. P. Starke, Elijah Swift, and the proposer.

Evaluation of a Determinant

E 955 [1951, 108]. *Proposed by C. L. Dunsmore, University of California, Los Angeles*

Evaluate the nth order determinant $|a_{ij}|$, where $a_{ij}=1/(i+j-1)!$.

Solution by J. T. Morse, Case Institute of Technology. We shall solve a slight generalization of the given problem. Let $I_{n,k}$ be the nth order determinant $|a_{ij}|$, where $a_{ij}=1/(i+j+2k-1)!$ for $k=0, 1, 2, \cdots$. Multiply the ith row by $i+2k$ and subtract the $(i-1)$th row for $i=n, n-1, \cdots, 2$. Factor $1-j$ from the jth column for $j=2, 3, \cdots, n$. Then

$$I_{n,k} = [(-1)^{n-1}(n-1)!/(n+2k)!]I_{n-1,k+1}$$

$$= (-1)^{n(n-1)/2}\prod_{r=1}^{n}[(r-1)!/(n+2k+r-1)!].$$

Also solved by F. Bagemihl, N. J. Fine, A. E. Livingston, F. D. Parker, F. A. E. Pirani, O. E. Stanaitis, L. M. Winer, the proposer, and an anonymous solver.

Bagemihl remarked that the proposed determinant was evaluated, incorrectly, by H. W. Segar, *The deduction of certain determinants from others of indeterminate form*, The Messenger of Mathematics, XXII (1892–1893), p. 67. Pirani showed that the problem is equivalent to finding the number of distinct plays in the game of linearized Chinese Chequers, invented by Professor K. F. Vomberg.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4453. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\frac{1^5}{e^{\pi}+1}+\frac{3^5}{e^{3\pi}+1}+\frac{5^5}{e^{5\pi}+1}+\cdots=\frac{31}{504}.$$

4454. *Proposed by R. C. Lyndon, Princeton University*

The "Burnside problem" for semi-groups without cancellation can be given the following form. Let $S(n, m)$ be the set of all words, or finite sequences, formed from an alphabet of m letters, which have the property that no (non-empty) part is repeated as many as n consecutive times. Show that, for $m, n>1$, and with the single exception $m=n=2$, the set $S(n, m)$ is infinite.

4455. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

If

$$\left[\frac{f(x)}{g(x)}\right]'=\frac{f'(x)}{g'(x)},$$

solve for $f(x)$ in terms of $g(x)$, and for $g(x)$ in terms of $f(x)$. (This is an extension of one of the problems proposed for the 1951 William Lowell Putnam Prize

Competition.)

4456. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If through the vertices A, B, C, D of a tetrahedron, parallel planes are drawn cutting a given line L in points A_2, B_2, C_2, D_2, and if A_1, B_1, C_1, D_1 are the points in which the lines AA_2, BB_2, CC_2, DD_2 cut the planes BCD, CDA, DAB, ABC, then

$$\frac{AA_2}{AA_1}+\frac{BB_2}{BB_1}+\frac{CC_2}{CC_1}+\frac{DD_2}{DD_1}=2.$$

4457. *Proposed by Israel Halperin, Queens University, Kingston, Ontario*

If

$$\sum_{m=1}^{\infty}|b_m|<\infty \quad \text{and} \quad \lim_{t\to 0}\sum_{m=1}^{\infty} b_m \cos\frac{1}{mt}=0,$$

show that every $b_m=0$.

SOLUTIONS

Number of Integers in One Random Set Greater than the Maximum of Another

4377 [1950, 41]. *Proposed by Paul Brock, Reeves Instrument Company, New York City*

Consider two non-decreasing sequences of n positive integers, $(n<M)$, the integers being chosen at random. What is the most probable number of integers in the sequence containing the maximum integer, each of which is larger than the maximum of the second sequence?

Solution by the Proposer. Given two sets,* each of n positive integers chosen at random. Assume an upper bound M on the values chosen, and a number $k \leqq M$. The probability that the first set has at least one element equal to k, and no element greater than k is

$$\left(\frac{k}{M}\right)^n-\left(\frac{k-1}{M}\right)^n.$$

The probability that the second set has i $(i>0)$ elements greater than k and $n-i$ elements not greater than k is

$${}_nC_i\left(\frac{k}{M}\right)^{n-i}\left(1-\frac{k}{M}\right)^i.$$

The probability of both events simultaneously is

* There seems to be no significance to the requirement that these sets be non-decreasing sequences.

$$2^nC_i\left(\frac{k}{M}\right)^{n-i}\left(1-\frac{k}{M}\right)^i\left\{\left(\frac{k}{M}\right)^n-\left(\frac{k-1}{M}\right)^n\right\},$$

where the possibility of interchanging the sets has been taken into account. Now the total probability of i integers in one set greater than the maximum of the other is the sum of this last probability taken over all k:

$$P_M(i) = 2^nC_i\sum_{k=1}^{M-1}\left(\frac{k}{M}\right)^{n-i}\left(1-\frac{k}{M}\right)^i\left\{\left(\frac{k}{M}\right)^n-\left(\frac{k-1}{M}\right)^n\right\}.$$

Let $k/M=x$, $\Delta k/M=\Delta x$, where $\Delta k=1$, and we have

$$P_M(i) = 2^nC_i\sum_{x=1/M}^{1-1/M}x^{n-i}(1-x)^i\left\{x^n-(x-\Delta x)^n\right\}.$$

We now let $M\to\infty$, eliminate higher order differentials, and replace summation by integration:

$$P(i) = 2n^nC_i\int_0^1 x^{2n-i-1}(1-x)^i dx.$$

Upon replacing the definite integral by $\beta(2n-i, i+1)$ we have finally, since n and i are integers,

$$P(i) = {}^nC_i/{}^{2n-1}C_i.$$

We note in passing that the probability of the two sets having the same maximum is

$$\sum_k\left\{\left(\frac{k}{M}\right)^n-\left(\frac{k-1}{M}\right)^n\right\}^2,$$

which approaches 0 as $M\to\infty$. As a check it is easy to show that $\sum_1^n P(i)=1$.

Since $P(1)=n/(2n-1)>\frac{1}{2}$, the value of i most likely to occur is 1. If "most probable value" is interpreted to mean the expected value, the answer is given by $\sum_1^n iP(i)$ which turns out to be $2n/(n+1)$.

Also solved by Roger Lessard, and J. G. Millar.

Laws of Attraction

4381 [1950, 119]. *Proposed by R. J. Walker, Cornell University*

The inverse square law has the property that the attraction, at an external point, due to a sphere of uniform density is the same as if the sphere were concentrated at its center. Are there any other laws of attraction which have this property?

Solution by J. L. Ericksen, University of Indiana. By choosing coördinates properly we may assume that the sphere is of radius a with center at the origin and that the attracted particle has coördinates $(0, 0, z)$ where $z>a\geqq 0$. Then,

with proper choice of units, we look for a continuous function $f(r)$, where r denotes distance, such that

$$(1) \qquad a^2 \int_0^{\pi} \int_0^{2\pi} \{(a \cos \theta - z)/r\} f(r) d\phi \sin \theta d\theta = -4\pi a^2 f(z)$$

where $r^2 = a^2 + z^2 - 2az \cos \theta$. This is easily reduced to the equivalent condition

$$I = \int_{z-a}^{z+a} (a^2 - z^2 - r^2) f(r) dr = -4az^2 f(z).$$

We have then

$$(2) \qquad \begin{aligned} \partial I/\partial a &= f(z+a)(-2az - 2z^2) + f(z-a)(2az - 2z^2) \\ &\quad + 2a \int_{z-a}^{z+a} f(r) dr = -4z^2 f(z), \end{aligned}$$

$$(3) \qquad \begin{aligned} \partial I/\partial z &= f(z+a)(-2az - 2z^2) - f(z-a)(2az - 2z^2) \\ &\quad - 2z \int_{z-a}^{z+a} f(r) dr = -8azf(z) - 4az^2 f'(z). \end{aligned}$$

Thus $f'(z)$ and, similarly, $f''(z)$ exist and are continuous. Eliminating the integral between (2) and (3), we have

$$(4) \qquad 2a^2 z f'(z) = f(z+a)(a+z)^2 + f(z-a)(z-a)^2 - f(z)(4a^2 + 2z^2).$$

If this is differentiated twice and then a is allowed to approach zero there results

$$z^2 f''(z) + 4zf'(z) - 2f(z) = 0$$

which has the solution $f(z) = Az + Bz^{-2}$, which satisfies the required condition (1). Thus the most general solution is $f(r) = Ar + Br^{-2}$.

Also solved by F. G. Fender, J. W. Green, Frank Herlihy, M. S. Klamkin, Roger Lessard, E. H. Sondheimer, A. E. Taylor, and the proposer.

Editorial Note. I. N. Sneddon and K. C. Thornhill, in the *Proceedings of the Cambridge Philosophical Society*, 1949, p. 318, treat a more general problem which reduces to the present discussion for the special case $\lambda = 0$. However, the limit as $\lambda \to 0$ was incorrectly taken with the result that the direct first power law was overlooked and the Newtonian potential was the only result found.

Klamkin notes that only the inverse square law provides that the force exerted on a particle inside a hollow sphere is zero. Herlihy notes that under the direct first power law an arbitrary body attracts as if its mass were concentrated at its center of gravity.

Modified Harmonic Series

4384 [1950, 120]. *Proposed by H. F. Sandham, Trinity College, Ireland*

In the following modification of the harmonic series there is a change of sign after the reciprocal of each square. Prove that

$$\frac{1}{1}-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{10}-\frac{1}{11}-\frac{1}{12}-\cdots$$

$$=\sum_{n=1}^{\infty}\frac{\pi}{\sqrt{n}\sinh\pi\sqrt{n}}.$$

Solution by the Proposer. In his solution of E 824 [1949, 264] Herzog gives necessary and sufficient conditions for the convergence of a general series of the first type. Let us write

$$f(x)=\frac{1}{1+x}-\frac{1}{2+x}-\frac{1}{3+x}-\frac{1}{4+x}+\frac{1}{5+x}+\frac{1}{6+x}+\cdots,\ x\geqq 0.$$

An application of Herzog's argument shows that

$$\frac{1}{1+x},\quad \frac{1}{2+x}+\frac{1}{3+x}+\frac{1}{4+x},\quad \frac{1}{5+x}+\frac{1}{6+x}+\cdots,\ \cdots$$

is a decreasing sequence. It follows easily that the series for $f(x)$ is convergent. We note, too, that $0<f(x)<1/(1+x)$, and that therefore as $x\to\infty$, $\lim f(x)=0$.

Now*

$$f(x-1)-f(x)=\frac{1}{x}-\frac{2}{x+1}+\frac{2}{x+4}-\frac{2}{x+9}+\cdots=\frac{\pi}{\sqrt{x}\sinh\pi\sqrt{x}}$$

for $x\geqq 1$. Letting x have integral values and summing from 1 to ∞ gives the result required.

Also solved by N. J. Fine and O. E. Stanaitis.

Cycles of n-digit Binary Integers

4385 [1950, 188]. *Proposed by P. Ungar, University College, London, England*

The three-digit sections of the sequence 1110001011 represent all three-digit numbers in the binary system exactly once each. For a given positive integer n an analogous sequence is obtained in the following manner: write down n 1's to begin with, and in each subsequent place write 0 unless the n-digit section thus completed occurs previously, in which case put 1. Show that the resulting sequence of 2^n+n-1 digits has the same property as the case $n=3$ cited at the outset.

Solution by Roger Lessard, École Polytechnique, Montreal, Canada. Let a_t stand for the n-digit binary number whose last digit occupies the tth place in

* Whittaker-Watson, *Modern Analysis*, 7.4, ex. 5.

the proposed sequence, and let $a_{\bar{i}}=a_i-1$. Note that, according to the construction rule, the occurrence of an odd $a_{\bar{i}}$ implies the previous occurrence of a_i. If, now, duplicate numbers occur in the sequence, let a_i be the first; that is

$$a_i = a_j, \qquad i < j.$$

As a_i is odd (a_j cannot be even by the construction rule) there exists a k such that

$$a_k = a_{\bar{i}} \qquad k < i$$

unless a_i is the first number of the sequence, $a_i=a_n=2^n-1$. Then a_k, a_i, a_j differ only in their last digit and therefore the numbers a_{k-1}, a_{i-1}, a_{j-1} have the same $n-1$ last digits; whence two of them are the same, contrary to the hypothesis.

It follows that the first number to repeat itself* is $a_n=a_j=2^n-1$. This last result implies

$$a_{j-t} \equiv 1 \pmod 2, \qquad 0 \leqq t < n.$$

As a_{j-1} is odd $a_{\overline{j-1}}$ occurs before it. In other words:

Corresponding to a_{j-1} there exist before a_j two numbers which have 0 as first digit followed by $n-2$ 1's, viz. $a_{j-1}=2^{n-1}-1$, $a_{\overline{j-1}}=2^{n-1}-2$.

The two numbers a_{j-2} and $a_{\overline{j-1}-1}$ have the same $n-1$ last digits so that they differ in the first digit only. As both are odd (same parity, a_{j-2} is odd) $a_{\overline{j-2}}$ and $a_{\overline{\overline{j-1}-1}}$ occur before them, so that:

Corresponding to a_{j-2} we have before a_j 2^2 distinct numbers having 0 as second digit followed by $n-3$ 1's. Two of them are $2^{n-2}-1$ and $2^{n-2}-2$.

As these numbers can be grouped in pairs having the same $n-1$ first digits the same reasoning can be repeated, giving:

Corresponding to $a_{j-k}(k=3, 4, \cdots, n-1)$ we have before $a_j 2^k$ distinct numbers having 0 as the kth digit followed by $n-k-1$ 1's. Two of them are $2^{n-k}-1$ and $2^{n-k}-2$.

We have then $2+4+\cdots+2^k+\cdots+2^{n-1}=2^n-2$ numbers before a_j. As these numbers include all numbers having 0 as a digit at any place except the last, the two remaining numbers of the 2^n numbers are 2^n-1 and 2^n-2, that is a_n and a_{n+1}. It follows that all 2^n numbers occur in the sequence before the first number a_n repeats, so that if the sequence is terminated at a_{j-1} we have each of the 2^n numbers exactly once.

Also solved by the proposer.

Editorial Note. The following statement of the problem has some advantages:

Let $a_1=2^n-1$, and form the sequence of n-digit binary integers, $a_1a_2a_3, \cdots$, where

$$a_{i+1} \equiv 2a_i \pmod{2^n}$$

* We do not at this stage consider the sequence terminated at 2^n+n-1 digits.

unless for some $i<j$, $a_i \equiv 2a_j$. In that case put

$$a_{j+1} \equiv 2a_j + 1 \pmod{2^n},$$

Then every n-digit integer will appear exactly once among the first 2^n members of the sequence.

A moment's reflection shows that, with the above rule of construction, if the particular integer 2^n-1 is not at the beginning it must be preceded by $2^{n-1}-1$ and 2^n-2 must occur earlier. But the immediate predecessor of 2^n-2 must be either 2^n-1 or $2^{n-1}-1$. Hence a necessary condition that the prescribed construction shall produce all n-digit integers is that the initial number be 2^n-1 or 2^n-2. The sequence for $a_1=2^n-2$ is the same as that for $a_1=2^n-1$ except that the number 2^n-1 is transposed from the left end to the right. The condition is therefore also sufficient.

D. D. Wall contributes the following notes. In the scale of notation r, a sequence of r^n digits which when interpreted cyclically contains each of the r^n different sequences of n digits as a consecutive subsequence—and hence each one once and only once—is called a cycle. N. G. de Bruijn (*Nederlandsche Akademie van Wetenschappen, Proceedings*, 1946, pp. 758–764) has shown that in base 2 the number of cycles is

$$2^{(2^{n-1}-n)}.$$

I. J. Good and D. Rees (*Journal of the London Mathematical Society*, 1946, pp. 167–172) have given existence proofs that such cycles exist for every scale of notation. The problem is also of interest in relation to normal numbers as defined in Hardy and Wright, *Theory of Numbers*.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Introduction to Algebraic Geometry. By W. G. Welchman. New York, Cambridge University Press, 1950. 10+349 pp. $4.50.

In writing this text the author had in mind a university course which would lead "as rapidly as possible to the study of configurations, loci and transformations in space of three, four and five dimensions." In order to facilitate progress, it seemed "desirable to apply to the more elementary problems the types of reasoning that are used in advanced work, instead of employing methods that are not capable of extension." Hence, we have a treatment of the

theory of conics which is "as much concerned with the development of technique as with the proving of theorems, and for this reason a theorem will often be proved by several different methods."

The particular algebraic geometry, to which this volume is an introduction, may be more exactly entitled "The Algebraic Geometry of the Field of Complex Numbers" in which the elements in a space of any number of dimensions are defined by ordered sets of complex numbers, but in which the only curves, surfaces and other constructs to be discussed are those which may be defined by algebraic equations whose coefficients are complex numbers and whose variables are complex variables. A theorem may be proved by non-algebraic methods, but the theorem itself must be a statement about algebraic equations. If we think of projective geometry as that part of algebraic geometry which may be deduced from the study of linear transformations, then nearly all of the geometry in this book is projective.

The foundations necessary for the later developments are established in the first three chapters (98 pages) and some of the topics discussed are the following: linear dependence, linear systems, ratio sets, polynomial equations, freedom of algebraic systems, geometric constructs in the plane, algebraic correspondence and rational systems of freedom one. The definition of a space of n dimensions is followed by quite detailed discussions of spaces of one, two and three dimensions. Duality and cross-ratio are introduced early and throughout the entire volume they play a very important role. Naturally enough many of these basic topics are treated abstractly and often briefly. Lest the beginner become discouraged, the author suggests that he read these chapters lightly at first and proceed with the theory of conics which follows. The frequent cross references given will gradually strengthen the reader's understanding of the foundations.

Chapter IV (The conic) deals with Joachimsthal's equation (name not given), parametric representation, cross-ratio on a conic, involution on a conic, polar theory and pencils of conics. Proofs are given for the conic locus and for its dual, the conic scroll. Chapter V (Configurations) considers the Desargues figure, the quadrangle and quadrilateral and the corresponding point and line pencils of conics. Chapter VI (Metrical geometry) introduces projective meanings for metrical terms and shows that in many cases the theorems of algebraic geometry reduce to theorems of Euclidean geometry. In Chapter VII (Homographic ranges on a conic) many of the theorems considered are essentially theorems about values of a parameter and, therefore, they may be stated as theorems about homographies in the elements of an arbitrary rational system of freedom one.

In Chapter VIII (Two conics. Reciprocation of one conic into another. Particular cases) two proper conics may have four distinct common points, simple, three-point, four-point or proper double contact. In each case standard forms for the equations are obtained and the reciprocation of a conic into a conic is investigated. In Chapter IX (Two conics. Apolarity) two proper conics determine certain conic loci and scrolls whose properties lead to the

consideration of the system of triangles inscribed in one conic and circumscribed to another. Linear systems of conic loci and conic scrolls are introduced. Chapter X (Two-two correspondences) discusses symmetric (2, 2) correspondences in order to derive properties of two conics leading to a proof of Poncelet's Theorem.

In Chapter XI (Application of matrix algebra) elementary matrix notation is introduced as an aid in the discussion of the configuration determined by two proper conics. Finally, Chapter XII (Invariants and covariants) deals with certain projective invariants and covariants which tend to unify the subject matter of the entire volume.

Of the dozen titles to which the author has referred, the most important is Robson's Introduction to Analytical Geometry. It is evident that great care and effort have gone into the writing and printing of this volume. However, the reviewer does feel that an index would increase the value of the book for reference purposes and that some well chosen problems would clarify and stimulate the reader's thinking.

C. H. YEATON

Analytic Geometry and Calculus. H. M. Gay. New York, McGraw-Hill, 1950. vii+485 pages+Tables, Answers, and Index. $5.00.

Analytic Geometry and Calculus. L. M. Kells. New York, Prentice-Hall, 1950. iii+552 pages+Appendix, Answers and Index. $4.75.

Elements of Mathematical Analysis, S. E. Urner-W. B. Orange. Boston, Ginn and Company, 1950. xi+506 pages+Appendix, Answers, and Index. $4.00.

These three books are written to provide college students an early introduction to the calculus, developing analytic geometry along side the calculus and introducing integral calculus before considering the derivatives of the logarithmic, exponential, or trigonometric functions. The Gay and the Kells assume that all necessary preparation has been made in algebra and trigonometry; the Urner-Orange though written for students with a background of intermediate algebra and trigonometry assumes a "considerable workout in the algebraic techniques will still be needed" and in trigonometry the authors "decided not to take anything for granted but to supply all basic subject matter."

The Urner-Orange is one of the "unified" mathematics texts and the authors state that the departure from "compartment" presentation has increased the general serviceability of the subject. The book starts with three chapters on functions in which are discussed graphs, formulas, and the rates of change of such functions as ax^2, the quadratic function, ax^3, and ax^4. Here the h notation is employed rather than the Δx and limits are found intuitively. In the next chapter, the ideas of limits and derivatives are presented with the customary theorems on the derivatives of products, quotients, *etc.* The appendix gives a further discussion of limits and continuity. Next comes a chapter on applica-

tions. The list of problems on maxima and minima is quite extensive, as in most calculus books, some being starred as more difficult. In the next chapter, integration is introduced but is limited to areas, a few volumes, and the equations of curves given by dy/dx equal to a polynomial in x.

Having gone thus far in the calculus, analytic geometry is developed through the conic sections. Then on to the calculus of the other functions where very little integration is given with logarithmic and exponential functions, none with trigonometric functions. These authors, in a course of five hours a week for a year, give calculus up to the formal integration of algebraic functions, including analytic geometry, polar coordinates, complex numbers, solution of equations, determinants, with the trigonometry and integration as mentioned above. The final two chapters, formal integration and applications, are used by these authors in a course in a second year, or the entire book is covered in three hours a week for two years. The book does not include space coordinates, partial derivatives, multiple integration, series, or differential equations. It has material to challenge the best of students and ample drill problems to develop technique.

The Gay starts with analytic geometry through conic sections, with the purpose of "giving enough work with coordinates to make graphs and common curves familiar, so that they may be used readily for illustrative purposes in the calculus." Included is a brief discussion of functions and functional notation. The other parts of analytic geometry are presented as needed. The calculus is introduced in a discussion on limits, with problems on the limits of fractions, then a definition of a derivative, followed by a physical interpretation of the derivative as a rate and a list of problems involving velocities, accelerations, *etc.* Then a chapter on such theorems as the derivative of a product, a quotient, . . . followed by a chapter on applications, ending with a substantial list of word problems on maxima and minima.

Integration is introduced before the study of trigonometric, logarithmic, or exponential functions. The applications are limited to areas and problems involving the distance an object moves. Problems involving force on submerged surfaces, work against gravitational forces, and moments appear only in the last third of the book after formal integration. Included in this book are chapters on series and differential equations through linear with constant coefficients. Of the three texts considered in this review, this one follows most closely the conventional in the exercises. Included however are "thought" problems as well as those that develop techniques.

The Kells, like the Gay, starts with analytic geometry but leaves conic sections to a later chapter. The first calculus is a chapter on limits including limits of a sequence and of a variable, special limits, theorems relating to limits, and continuity. The idea of a rate of change is introduced through velocity as the limit of an average velocity. Next a derivative is defined, followed by the theorems for the differentiation of products, quotients, *etc.* Then there follows a chapter on the geometric application of the derivative and another chapter on

velocities, accelerations, and rates.

Next comes integration, in three chapters, showing such applications as areas, volumes by slices and of revolution, length of arc, moments, fluid pressures, work, the trapezoidal and Simpson's rules. This book has a chapter on Vectors, "which requires their differentiation, interpretations of them as velocities, accelerations." Also included are linear differential equations with constant coefficients and applications.

The Kells does not follow the convention of having the exercises set in a smaller size type than the body of the text which, to this reviewer, makes more difficult a quick review and the making of assignments for classes. The other books are provided with answers to the odd numbered problems with a few exceptions in the case of the Gay. The Kells apparently gives most of the answers except to certain drill problems but no statement as to the system for giving answers appeared in the usual places. All three books are testimonials to the excellent work of compositors and publishers.

E. A. WHITMAN

NEW BOOKS RECEIVED

A New Theory of Gravitation. By Jakob Mandelker. New York, The Philosophical Library, 1951. 25 pp. $4.75.

Abstract Algebra, Vol. I. By Nathan Jacobson. D. Van Nostrand Company, New York. $5.00. 1951.

Brief Course in Analytics. M. A. Hill, Jr. and J. B. Linker. New York, Henry Holt, 1951. xi+224 pp.

College Algebra. By H. L. Rietz, A. R. Crathorne and J. W. Peters. New York, Henry Holt, 1951. xv+387 pp.

Fourier Transforms. By I. N. Sneddon. New York, McGraw-Hill, 1951. xii +542 pp. $10.00.

Analytic Geometry, Second Edition. By J. W. Cell. New York, John Wiley and Sons, 1951. xii+326 pp. $3.75.

Algebra for Commerce and Liberal Arts. By Bettinger and Dwyer. New York, Pitman Publishing Corporation, 1951. xi+225 pp. $3.00.

Quantum Mechanics of Particles and Wave Fields. By Arthur March. New York, John Wiley and Sons, Inc., 1951. 10+292 pp. $5.50.

Tables Relating to Mathieu Functions. National Bureau of Standards, The Computation Laboratory. New York, Columbia University Press, 1951. xviii +278 pp.

Using Your Mind Effectively. By J. L. Mursell. New York, McGraw-Hill, ix+264 pp. $3.00.

Teaching the New Arithmetic, Second Edition. G. M. Wilson. New York, McGraw-Hill, 1951. xiv+483 pp. $4.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

All material for this department should be sent to Professor H. D. Larsen, Albion College, Albion, Michigan.

CLUB REPORTS, 1950–51

Mu Alpha Theta, Yeshiva College

The Yeshiva College *Mathematics Club, Mu Alpha Theta*, devoted its program to lectures, delivered by the students, on various mathematical topics, including the following:

Abstract algebra; *Construction of the number system*; *Mathematical logic*, a series of four lectures by Mr. Rosenfeld

Vector algebra, by Mr. Wenger

Mathematical inversion, by Mr. Solomon

Probability, two lectures by Mr. Lieberman and Mr. Hertzberg

Elementary topology, by Mr. Hellerstein

Stieltjes integration, by Mr. Haber

Cryptography, by Mr. Levy

Infinite series, two lectures by Mr. Hellerstein

Special relativity, by Mr. Landau

The prime pair theorem, by Mr. Klein.

Other activities included a number of parties, and several informal discussions on miscellaneous problems in recreational and educational mathematics.

The present officers of the club are: President, Irwin Wenger; Vice-President, Simon Hellerstein.

The Mathematics Society, The Cooper Union

The following talks were presented to the *Mathematics Society* of The Cooper Union during 1950–51:

Solution of algebraic equations, by Peter Redmond

Elementary number theory, by Alan Berndt

Theory and use of the slide rule, by Jerry Davis

Calculus of finite differences, by Walter Kahn

Coaxial circles, by Peter Redmond

Inversion, by Peter Redmond

Introduction to matrices, by Irving Lowe

Nomographs, by Walter Kahn

Iteration, by Walter Kahn

Higher plane curves, by Harry Schwartzlander

Laplace transform, by Peter Redmond

Symbolic logic, by Arthur Robinson.

The officers for 1951–52 are: President, Alan Berndt; Vice-President and Treasurer, Harry Schwartzlander; Secretary, Walter Pecota; Faculty Adviser, Prof. J. N. Eastham.

Kappa Mu Epsilon, Hofstra College

The *New York Alpha* chapter of *Kappa Mu Epsilon*, during the academic year 1950–51, held monthly meetings with talks by faculty members, students, and alumni members. The topics and speakers were:

Report on the International Mathematics Congress, by Dr. E. R. Stabler
Probabilities, by Peter Hinrichs
The wineglass problem, by Mr. Frank Hawthorne
Continued fractions, by Peter Marshall
The normal law of error, by Dr. Mildred Dean
The trisection problem, by Richard Jaeger
Laplace transforms, by Sam Reynolds
Paradox, by Mr. W. Lysle Marshall
Selective sequence electronic calculator, by Ruth Mayer
Mathematics in the field of teaching, by Gertrude Decker
Mathematics in musical scales, by Robert Blasch
Probability—geometrical methods, by Kenneth Feldmann
The ladder problem, by Alexander Basil and Lee Dunbar.

Mr. Preston R. Bassett, President of Sperry Gyroscope Co. spoke on *The inventor discovers mathematics* at the Tenth Annual Initiation Banquet. Leaders of Long Island industries were invited as guests for the occasion. Mr. Bassett was initiated as an honorary member of the *New York Alpha* chapter.

Each year the *New York Alpha* chapter presents an award to the best student of freshman mathematics. The winner, Sue Rae Waldman, was selected on the basis of her grades in freshman mathematics and a two-hour competitive examination.

Miss Jane Brandt received a Fulbright scholarship for study in Biological Statistics in England for the academic year 1951–52.

The chapter has sponsored student help sections where *Kappa Mu Epsilon* members gave their time to help the weaker students in mathematics.

Eight of the members attended the national convention held in Springfield, Missouri and reported to the group at the last regular meeting of the year. Miss E. Marie Hove and Dr. L. F. Ollmann were reelected National Secretary and National Treasurer, respectively, at the convention.

Other activities included a square dance and Christmas party, and a picnic.

The newly elected officers to serve for 1951–52 are: President, Robert Blasch; Vice-President, Richard Lamm; Secretary, Marjorie Liers; Treasurer, Louis Bronzo; Historian, Mary Pawelko; Corresponding Secretary, Dr. Mildred Dean; Sponsor, E. Marie Hove.

Mathematics Club, Wellesley College

The Wellesley College *Mathematics Club* had for its 1950–51 program a series of lectures pertaining to various phases of mathematics. The lectures and speakers were:

Job possibilities for mathematics majors, by Mrs. R. L. Bishop of the Wellesley

Placement Office

Mathematics of philosophy, by Miss Onderdonk of the Wellesley Philosophy Department

The measurement theory, by Walter Rudin, Instructor at Massachusetts Institute of Technology.

The officers for 1950–51 were: President, Elizabeth Robinson; Vice-President, Ursula Loengard; Secretary, Barbara Kuehn; Treasurer, Eleanore Torstenson; Junior Executive, Marjorie Matsis; Sponsor, Helen G. Russell.

Mathematics Club, Indiana State Teachers College

The *Mathematics Club* of Indiana State Teachers College, Indiana, Pennsylvania had a very successful year, the high spot being a mathematical Christmas Party. The meetings, which proved to be very interesting consisted of the following talks:

Non-Euclidean geometry, by John Toman

Use of the slide rule, by Donald Leffler

Recreational mathematics, by Barbara Johnson, Leonora Murray, and Shirley Smith

Problems of a student teacher in mathematics, by Nancy Wallace, H. C. McKowen, and Robert Whistner

Applications of mathematics in road construction, by C. L. Playfoot, Pennsylvania Department of Highways.

Officers for the year 1950–51 were: President, James Norman Cornell; Secretary, Shirley Reynolds; Sponsors, Dr. Joy Mahachek and Dr. I. L. Stright.

Pi Mu Epsilon, University of Nebraska

Four meetings were held by the *Nebraska Alpha* chapter of *Pi Mu Epsilon* during 1950–51, in addition to social meetings. The program consisted of the following talks:

An application of the solution of a special recursive series, by Dr. H. B. Ribeiro

A method of quality production control, by J. W. Adams

Biology and mathematics, by Prof. B. H. Burma

Movement of pressure systems, by Prof. L. K. Jackson.

The new officers for 1951–52 are: Director, George Cobel; Vice-Director, Bruce Emmons; Secretary, Donna Grueber; Treasurer, Winfred Zacharias.

Mathematics Club, Kansas State Teachers College, Pittsburg

The activities of the *Mathematics Club* of the Kansas State Teachers College at Pittsburg included two picnics and the following programs:

History of calendar forms, by Joe Butler

Mathematical recreations, by Tom Clark

Magic squares, by Prof. I. G. Wilson

The language of mathematics, by James Pike
Spherical trigonometry, by William Brumbaugh
Celestial navigation, by Prof. R. W. Hart
The squirrel cage slide rule, by Robert Green, a paper also given at the National Convention of *Kappa Mu Epsilon* in Springfield, Missouri.

Morse Mathematics and Physics Club, University Heights Branch of New York University

The *Morse Mathematics and Physics Club* of University Heights held bi-weekly meetings during the academic year 1950–51. Lectures presented were:
Discharges in gases, by Prof. L. H. Fisher
Approximate homomorphisms, by Prof. H. N. Shapiro
Matric methods in the solution of systems of linear differential equations, by Prof. Avron Douglis
Nuclear emulsions, by Arthur Beiser
Cosmic rays, by Prof. Serge Korff
Special methods in the evaluation of definite integrals, by Prof. P. L. Thorne
Random noise, by Prof. Yardley Beers
Coding of problems for electronic computers, by Herbert Goertzel
Matrices as applied to mechanics, by Mr. L. Leitner
Farey series, by Dr. Sholom Arzt
The calculus of variations, by Charles Solky.
Bi-weekly seminars were also held, at which Harold Bohr's work, *Almost Periodic Functions*, was discussed.
The officers for the year 1950–51 were: President, Charles Solky; Vice-President, Burton Wendroff; Secretary, Herbert Kranzer.

Pi Mu Epsilon, St. Louis University

The *Missouri Gamma* chapter of *Pi Mu Epsilon*, during the academic year 1950–51 held three regular meetings, including business meetings and elections, besides the annual initiation banquet. The programs consisted of lectures on the following topics:
Foundations of mathematics, by Bernard Derwort
Partial differential equations, by John Hoffschwelle
Line geometry, by Dr. Paolo Lanzano
Chance, by Dr. Paul Rider, Professor of Mathematics, Washington University.
The fifth Annual Prize Essay Contest open to undergraduate students only, was conducted by Mr. Alois Lorenz. The senior division was won by Miss Helen Fagan, a senior of Maryville College. The title of her paper was *The mind of Newton as reflected in the Principia*. Her award was D. E. Smith's *Source Book of Mathematics*. Miss Maureen Burke, a sophomore of Fontbonne College, won the junior award for her essay *Isaac Newton: his life and works*. She was awarded E. T. Bell's *Men of Mathematics*.
James Krebs received the Chemical Rubber Publishing Co.'s *Book of Tables*

for being the freshman with the highest scholastic standing in mathematics. Roger Ahrens was given the Garneau Award of twenty-five dollars for being the highest ranking senior majoring in mathematics.

Seventy-five new members were inducted into the chapter at the spring meeting, bringing the total number of active members to two hundred and ninety-one and the total membership since the charter was granted to five hundred and fifty-three.

The Missouri Gamma News, the chapter bulletin, will be published during the summer.

The officers for 1950–51 were: Director, Joseph Santner; Vice-Director, Richard Kern; Secretary-Treasurer, Virginia Herre; Faculty Advisor and Permanent Secretary-Treasurer, Dr. Francis Regan.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

EDUCATIONAL TESTING SERVICE PSYCHOMETRIC FELLOWSHIPS

The Educational Testing Service is offering for 1952–53 its fifth series of research fellowships in psychometrics leading to the Ph.D. degree at Princeton University. Open to men who are acceptable to the Graduate School of the University, the two fellowships each carry a stipend of $2,375 a year and are normally renewable.

Fellows will be engaged in part-time research in the general area of psychological measurement at the offices of the Educational Testing Service and will, in addition, carry a normal program of studies in the Graduate School. Competence in mathematics and psychology is a pre-requisite for obtaining these fellowships. The closing date for completing applications is January 18, 1952. Information and application blanks will be available about November 1 and may be obtained from: Director of Psychometric Fellowship Program, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

THE TENSOR SOCIETY

A new mathematical organization, the Tensor Society, has been founded recently at Hokkaido University, Sapporo, Japan. Membership is open to all interested persons. The dues are $2.00 per year. The Society's journal, *Tensor, New Series*, will appear in two or three issues annually and will contain forty to eighty pages to the issue. *Tensor* will be devoted primarily to short, high grade

articles in the general field of vector and tensor analysis and certain domains of application such as mathmatical physics (including statics and dynamics), engineering, astronomy, geodesy, physiology, and philosophy. Application blanks for membership may be obtained from Professor A. Kawaguchi, Faculty of Science, Hokkaido University, Sapporo, Japan, or from Professor H. V. Craig, Department of Applied Mathematics, University of Texas, Austin, Texas.

MATHEMATICS PRIZE COMPETITION AT LOS ANGELES CITY COLLEGE

Los Angeles City College held its first annual William B. Orange Mathematics Prize Competition for high school students of the Los Angeles City High Schools on May 25, 1951. The competition was inaugurated as a memorial to the late William B. Orange, formerly chairman of the Department of Mathematics of Los Angeles City College.

The competition consisted of a two hour written examination. Schools were permitted to send two teams of three students each. There were one hundred sixty-one participants from thirty-three high schools. The team prize was a bronze trophy which was won by Eagle Rock High School. Also, several individual prizes were awarded.

A second competition will be held next spring. The bronze trophy will be presented annually to the winning school.

PERSONAL ITEMS

Dr. J. A. Clarkson, head of the Department of Mathematics of Tufts College, has succeeded Professor J. R. Kline in the position of Executive Secretary of the Division of Mathematics, which was recently established by the National Research Council.

Professor J. J. Nassau, director of the Observatory, Case Institute of Technology, has been elected Treasurer of the American Astronomical Society.

Illinois Institute of Technology announces the following: Professor L. R. Ford will be absent from the campus on special assignments during the year 1951–52; Professor Haim Reingold is Executive Officer of the Department of Mathematics for the year; Assistant Professor S. S. Shu has been promoted to an associate professorship.

At New York State College for Teachers, Albany: Professor Harry Birchenough, chairman of the Department of Mathematics, has retired with the title of Professor Emeritus; Professor R. A. Beaver has been promoted to the chairmanship of the department.

University of New Hampshire reports: Associate Professor M. R. Solt has been promoted to a professorship; Mr. R. O. Kimball has been promoted to an assistant professorship; Dr. R. B. Davis and Dr. H. G. Rice have been appointed to assistant professorships; Mr. Frederic Cunningham has been appointed to an instructorship.

Mr. R. C. Ailara, who has been a graduate student at Illinois Institute of Technology, has accepted a position as Junior Systems Engineer at Chance

Vought Aircraft, Dallas, Texas.

Mr. P. M. Anselone, formerly a graduate student at the College of Puget Sound, has a position as mathematician at General Electric—Hanford Works, Richland, Washington.

Assistant Professor Winifred A. Asprey of Vassar College has been promoted to an associate professorship.

Mr. J. H. Braun, formerly a student at Illinois Institute of Technology, has a position as mathematical physicist at the Naval Gun Factory, Washington, D. C.

Mr. R. L. Brooks, previously a graduate student at Catholic University, has accepted a position as mathematician at the National Bureau of Standards, Washington, D. C.

Mr. L. P. Burton, who has been a graduate student at the University of North Carolina, is now at the University of California at Davis.

Assistant Professor R. C. Campbell of the United States Naval Postgraduate School has been promoted to an associate professorship.

Assistant Professor F. M. Carpenter, Colorado School of Mines, has been promoted to an associate professorship.

Mr. T. E. Cheatham, Jr., has been appointed to a research assistantship at Purdue University.

Mr. Carl Cohen of Cambridge Junior College has been appointed Lecturer at Wheaton College, Norton, Massachusetts.

Mr. K. J. Cohen, formerly a student at Reed College, is now a Rhodes Scholar at Queen's College, Oxford University, Oxford, England.

Dr. E. P. Coleman has accepted a position as mathematical statistician with the Hughes Aircraft Company, Culver City, California.

Mr. D. F. Coulter, Jr., formerly a teaching assistant at State College of Washington, has a position as mathematician at the United States Navy Mine Countermeasures Station, Panama City, Florida.

Mr. E. L. Eagle of the University of Tennessee has accepted a position as electromechanical engineer at Glen L. Martin Aircraft Company, Baltimore, Maryland.

Mr. James Eastcott, formerly a teaching fellow at the University of Michigan, is now a research scientist with the Defense Research Board, Ottawa, Canada.

Mr. R. P. Eisinger, previously a student at Swarthmore College, is enrolled in the College of Physicians and Surgeons, Columbia University.

Dr. H. Margaret Elliott of Washington University has been promoted to an assistant professorship.

Graduate Assistant P. G. Engstrom of Kansas State College is now at Augustana Theological Seminary.

Professor C. H. Forsyth of Dartmouth College has retired.

Assistant Professor Stanley Frankel of the Department of Applied Mathematics, California Institute of Technology, was at Victoria University of

Manchester, England, during the summer and was engaged in research for the British government.

Dr. L. E. Fuller of the University of Wisconsin has a position as senior development engineer with the Goodyear Aircraft Corporation, Akron, Ohio.

Assistant Professor R. W. Gardner of Olivet Nazarene College has been promoted to the position of Professor and Head of the Department of Mathematics.

Mr. R. N. Goss of the University of Tulsa has a position in the United States Navy Electronics Laboratory, San Diego, California.

Mr. H. W. Gould, formerly a student at the University of Virginia, is now in the United States Army.

Mr. E. P. Graney of the University of Notre Dame has been appointed Research Assistant at Willow Run Research Center, University of Michigan.

Associate Professor J. W. Green of the University of California at Los Angeles is at the Institute for Advanced Study.

Dr. W. T. Guy, Jr., previously a graduate assistant at California Institute of Technology, has been appointed Assistant Professor of Applied Mathematics and Astronomy at the University of Texas.

Assistant Professor J. J. Hayes of the University of Utah has retired.

Associate Professor P. W. Healy of Southwestern College has been appointed Assistant Professor of Mathematics and Astronomy at the University of New Mexico.

Dr. L. Aileen Hostinsky of Temple University has been promoted to an assistant professorship.

Associate Professor S. T. Hu of Tulane University delivered a series of lectures on Cohomology Theory of Topological Groups at the University of Michigan during the Summer Session of 1951.

Professor C. M. Huber, chairman of the Division of Mathematics and Business Education, James Ormond Wilson Teachers College, has been appointed Dean of Instruction, Montclair State Teachers College, New Jersey.

Assistant Professor W. J. Klimczak of the University of Rochester has been appointed to an assistant professorship at Trinity College, Hartford, Connecticut.

Mr. W. G. Koellner, formerly a student at New Jersey State Teachers College, is teaching at Hillside High School, Hillside, New Jersey.

Mr. Charles Kurland, previously a graduate student at the University of Buffalo, has accepted a position as research engineer, Locomotive Development Company, Alco Plant, Dunkirk, New York.

Associate Professor J. P. LaSalle of the University of Notre Dame is now in the United States Navy.

Dr. D. B. Lloyd has been appointed to an instructorship at James Ormond Wilson Teachers College.

Associate Professor N. H. Mewaldt of Northern State Teachers College, Aberdeen, South Dakota, has been promoted to a professorship.

Professor W. L. Miser, who has retired from the position of Head of the Applied Mathematics Department, Vanderbilt University, has accepted a position at McKendree College.

Dr. Deane Montgomery of the Institute for Advanced Study has been promoted to the position of Professor of Mathematics.

Associate Professor R. H. Moorman has been promoted to the position of Professor and Head of the Department of Mathematics of Tennessee Polytechnic Institute.

Mr. W. L. Murdock, formerly of Cornell University, has a position as consulting and research mathematician at General Electric Electronics Laboratory, Syracuse, New York.

Graduate Assistant J. D. Neff of Kansas State College has been appointed to a graduate assistantship at Purdue University.

Associate Professor Zeev Nehari of Washington University has been promoted to a professorship.

Mr. George Ogawa, previously a student at the University of Chicago, is now in the United States Army.

Mr. S. V. Parter, recently a student at Illinois Institute of Technology, has been appointed to a research assistantship at the Los Alamos Scientific Laboratory of the University of California.

Mr. M. F. Pollack of Stockton College is teaching now at the Union Linden High School, Linden, California.

Assistant Professor M. H. Protter of Syracuse University is a member of the Institute for Advanced Study during 1951–52.

Sister M. Virgilia, who has been a student at Notre Dame University, is now at Madonna College, Plymouth, Michigan.

Associate Professor T. H. Southard of Wayne University is at the Institute for Numerical Analysis.

Mr. A. L. Stamper, recently a student at Eastern Kentucky State College, is in military service.

Associate Professor J. E. Stewart of the University of Maine has been appointed Dean of Men.

Lecturer N. Y. Tang of the University of Washington has accepted a position with the Actuarial Department, Manhattan Life Insurance Company, New York City.

Associate Professor Alexander Tartler of Lafayette College has been promoted to a professorship.

Mr. R. E. Walters, formerly a graduate student at Kent State University, has a position as mathematician at Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Mr. M. E. White of Hunter College is in the United States Army.

Mr. W. F. Winn is Military Instructor at Camp Wood, New Jersey.

Mr. E. G. Wohlford, who has been teaching at the Iolani School, Honolulu, Hawaii, is now in the United States Army.

Associate Professor H. E. Wolfe of Indiana University has been promoted to a professorship.

Assistant Professor A. D. Ziebur of Oklahoma Agricultural and Mechanical College has been appointed to an instructorship at Ohio State University.

Mr. R. L. Adey, who was a student at Oregon State College, was killed in an automobile accident on May 12, 1951. The annual mathematics award of Oregon Beta Chapter of Pi Mu Epsilon has been named after him as a commemoration.

Professor Emeritus Abraham Cohen of Johns Hopkins University died on April 25, 1951. He was a charter member of the Association.

Miss Florence L. Munroe, who had retired from her position of Head of the Department of Mathematics, Northampton High School, Northampton, Massachusetts, died in September, 1949. She was a charter member of the Association.

Reverend Joseph Wilczewski, S.J., of the Mathematics Department of Marquette University, died on March 6, 1951. He had been a member of the Association for twenty-eight years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following seventy-six persons have been elected to membership by the Board of Governors on applications duly certified:

F. H. APPLEBAUM, Student, University of Pennsylvania, Philadelphia, Pa.

J. T. ARCHIBALD, Student, St. John's College, Booklyn, N. Y.

W. L. BAILY, Jr., Student, Massachusetts Institute of Technology, Cambridge, Mass.

MRS. BARBARA B. BLAIR, M.S. (Iowa) Part-time Instr., State University of Iowa, Iowa City, Iowa

I. L. BOSSLER, M.S. (Purdue) Grad. Assistant, Purdue University, Lafayette, Ind.

H. A. BOTT, Student, Illinois Institute of Technology, Chicago, Ill.

O. C. BRAUNE, Student, St. Mary's University, San Antonio, Tex.

ANITA BREDT, Student, University of Pennsylvania, Philadelphia, Pa.

DOROTHY C. BREYNAERT, Student, University of New Hampshire, Durham, N. H.

EVA BURKHALTER, M.S. (Oregon) Teacher, Klamath Union High School, Klamath Falls, Ore.

H. E. CHRESTENSON, M.A. (Washington S.C.) Teaching Assistant, State College of Washington, Pullman, Wash.

HELEN E. Core, M.S. (Michigan S.C.) Research Assistant, University of Michigan, Ann Arbor, Mich.

D. F. COULTER, JR., M.A. (Washington S.C.) Mathematican, U. S. Navy, Panama City, Fla.

D. O. DAVIDSON, B.S. (Oregon S. C.) 1833 5th Street, Manhattan Beach, Calif.

T. E. DEPKOVICH, B.S. (Indiana S.T.C.) Asst. Professor, College of Steubenville, Ohio.

JOAN DRAKE, Student, Seton Hill College, Greensburg, Pa.

D. G. DUNCAN, Ph.D. (Michigan) Asst. Professor, University of Arizona, Tucson, Ariz.

R. P. Eisinger, Student, Swarthmore College, Pa.

MILTON FISHMAN, B.S.E. (Michigan) Automobile Design Engineer, Detroit Arsenal, Centerline, Mich.

D. J. R. FOULIS, Student, University of Miami, Coral Gables, Fla.

MRS. HERTA T. FREITAG, M.A. (Columbia) Asst. Professor, Hollins College, Va.

H. N. GARBER, Student, University of Pennsylvania, Philadelphia, Pa.

S. H. GREENE, M.A. (Pennsylvania) Lieutenant, U. S. Army, Fort Bliss, Tex.

R. C. GUNNING, Student, University of Colorado, Boulder, Colo.

H. M. GURK, Student, University of Pennsylvania, Philadelphia, Pa.

SIMON HELLERSTEIN, Student, Yeshiva University, New York, N. Y.

J. A. HUMMEL, B.S. (C.I.T.) U. S. Army, Red Bank, N. J.

ELEANOR L. JENKINS, M.A. (North Carolina) Asso. Professor, Queens College, Charlotte, N. C.

C. H. JONES, 1178 King Street, Charleston, S. C.

M. W. KALLETT, A.B. (Buffalo) Teacher, North Park Junior High School, Lockport, N. Y.

MRS. ROSELLA KANARIK, Ph.D. (Pittsburgh) Lecturer, University of Southern California, Los Angeles, Calif.

ROSEMARY KANE, Student, Seton Hill College, Greensburg, Pa.

GENE KERNS, A.B. (Chico S.C.) P.O. Box 535, Susanville, Calif.

SIDNEY KISSIN, B.S. (Brooklyn C.) 2111 67 Street, Brooklyn, N. Y.

PATIENCE B. KLOPP, Student, Seton Hill College, Greensburg, Pa.

W. G. KOELLNER, Student, New Jersey State Teachers College, Upper Montclair, N. J.

T. J. LEE, 89–12 205 Street, Hollis, N. Y.

MILTON LEES, Student, University of California, Berkeley, Calif.

JULIUS LIEBLEIN, M.A. (Brooklyn C.) Mathematician, National Bureau of Standards, Washington, D. C.

J. L. LOCKER, Student, Alabama Polytechnic Institute, Auburn, Ala.

P. H. LORD, A.B. (Princeton) Lieutenant, U. S. Army, Fort Sill, Okla.

A. R. MATTHEWS, Student, Alabama Polytechnic Institute, Auburn, Ala.

R. B. MCMURDO, B.S.E.E. (U.S. Naval Academy) Instr., Montana State College, Bozeman, Mont.

E. A. MEANS, Student, Friends University, Wichita, Kans.

A. E. MILLER, Student, University of Pennsylvania, Philadelphia, Pa.

F. D. MILLER, A.M. (Michigan) Mathematician, Bell Aircraft Corporation, Niagara Falls, N. Y.

J. S. MORISON, S.B. (Chicago) 1445-C Stanford Street, Santa Monica, Calif.

D. E. MORRILL, B.A. (Mississippi) Grad. Fellow, University of Mississippi, University, Miss.

H. T. MUHLY, Ph.D. (Johns Hopkins) Asso. Professor, State University of Iowa, Iowa City, Iowa.

ZEEV NEHARI, Ph.D. (Hebrew U., Jerusalem) Professor, Washington University, St. Louis, Mo.

S. I. NEUWIRTH, M.S. (N.Y.U.) Bio-statistician, Schering Corporation, Bloomfield, N. J.

LEONARD NICHOLS, Student, New Jersey State Teachers College, Upper Montclair, N. J.

W. A. NICHOLSON, JR., Transmission Man, American Telephone and Telegraph Company, Charlotte, N. C.

J. L. O'DAY, Student, Gonzaga University, Spokane, Wash.

MARTIN ORR, M.A. (Columbia) Mathematician, Signal Corps Center, Ft. Monmouth, N. J.

F. D. PARKER, Ph.D. (Case) Asst. Professor, St. Lawrence University, Canton, N. Y.

MRS. LOIS K. PARRISH, Student, Eastern Kentucky State Teachers College, Richmond, Ky.

M. F. POLLACK, Master (Berlin) Teacher, Stockton College, Calif.

P. J. REDMOND, Student, Cooper Union,

New York, N. Y.

H. E. REINHARDT, M.A. (Washington S.C.) Teaching Assistant, State College of Washington, Pullman, Wash.

E. C. RICE, M.A. (George Peabody) Grad. Student, George Peabody College for Teachers, Nashville, Tenn.

JUDITH A. RICHMAN, Student, University of Pennsylvania, Philadelphia, Pa.

A. E. RICHMOND, Certificate (Multnomah) Instr., Multnomah College, Portland, Ore.

M. B. RITTERMAN, M.S. (N.Y.U.) Instr., Long Island University, Brooklyn, N. Y.

G. K. ROSENBLUM, Student, University of Pennsylvania, Philadelphia, Pa.

W. T. SANDLIN, A.B. (Marshall) Manager and Advertising Director, C. E. Sandlin & Son, Richmond, Ky.

J. J. SCHODERBEK, M.S. (Carnegie) Electronics Engineer, Glen L. Martin Company, Baltimore, Md.

M. J. SENDROW, Student, University of Michigan, Ann Arbor, Mich.

SISTER ST. THOMAS OF CANTERBURY, M.A. (Fordham) Teacher, Villa Maria Academy, Bronx, N. Y.

O. E. STANAITIS, Ph.D. (Würzburg) Asst. Professor, St. Olaf College, Northfield, Minn.

MARIE A. STEIN, Student, Adelphi College, Garden City, N. Y.

F. D. STROW, M.A. (Bowling Green) Head of Mathematics Department, Perrysburg High School, Ohio.

C. A. SWANSON, B.A. (British Columbia) Grad. Student, University of British Columbia, Vancouver, B. C.

HERMAN VON SCHELLING, Ph.D. (Berlin) Biomathematician, Naval Medical Research Laboratory, U. S. Submarine Base, New London, Conn.

F. J. WALL, B. S. (Sul Ross S.C.) Mathematical Analyst, University of California, Los Alamos, N. M.

J. F. WILLIAMS, M.A. (Tennessee) Asst. Professor, Memphis State College, Tenn.

JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The thirteenth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of San Francisco, San Francisco, California, on Saturday, January 27, 1951. Mr. S. A. Francis, Chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was seventy-six including the following forty-five members of the Association: H. L. Alder, H. M. Bacon, G. A. Baker, Alice K. Bell, M. T. Bird, A. C. Burdette, M. A. Dernham, Brother Dominic, Roy Dubisch, Hazel E. Eggett, E. B. Eilertsen, G. C. Evans, Ruth A. Fish, S. A. Francis, N. S. Free, C. M. Fulton, L. C. Graue, J. G. Herriot, Marjorie L. Hoffman, D. W. Hullinghorst, V. F. Ivanoff, Helene G. Kusick, D. H. Lehmer, Sophia L. McDonald, W. H. McKenzie, A. B. Mewborn, E. D. Miller, H. S. Moredock, F. R. Morris, W. H. Myers, Andrewa R. Noble, C. D. Olds, George Pólya, Raphael M. Robinson, Kathryn B. Rolfe, Sister Madeleine Rose, Mary V. Sunseri, Irving Sussman, Gabor Szegö, M. J. Vitousek, K. J. Waider, L. A. Walker, Elizabeth Weinstock, Anna P. Wheeler, A. R. Williams.

The following officers were elected for the coming year: Chairman, Professor D. H. Lehmer, University of California; Vice-Chairman, Professor C. D. Olds, San Jose State College; Secretary-Treasurer, Mrs. Marjorie L. Hoffman, San Mateo Junior College.

By invitation of the Section, Professor J. G. van der Corput of the University of Amsterdam and visiting Professor at Stanford University delivered an address during the morning session.

The following papers were presented:

1. *Integral solutions of* $x^2+y^2=z^2$ *in quadratic fields*, by Professor C. D. Olds, San Joseph State College.

Every Diophantine equation whose history is given in Dickson's *History of the Theory of Numbers* admits of a generalized treatment in algebraic fields of the second, third, and higher degrees. The treatment of many such equations is well within the reach of first year graduate students. Using the above equation as an example, the author indicates how by considering special cases the student can be encouraged to attempt the general solution. In this case general formulas for x, y, and z can be obtained. It is also possible to prove that in every quadratic field there is an infinity of solutions through algebraic integers x, y, z belonging to the same field $R(\sqrt{i})$.

2. *On plausible reasoning*, by Professor George Pólya, Stanford University.

The speaker discussed historical examples of plausible reasoning in mathematical matters (by Euler, Descartes, Lord Rayleigh) and showed how these examples lead naturally to certain patterns of plausible inference which we seem to follow also in everyday matters, in court proceedings, in experimental science, etc. The simplest pattern of this kind is discussed in the speaker's booklet "How to Solve It," 5th printing, pp. 220–224. The title of the talk was the same, and the content almost the same, as that of an invited address given by the speaker at the International Congress of Mathematicians in Cambridge, on September 2, 1950.

3. *The second pearl of the theory of numbers*, by Professor J. G. van der Corput, University of Amsterdam, introduced by the Secretary.

If the number of positive elements $\leqq h$ of a sequence A, formed by integers $\geqq 0$, is $\geqq \alpha h$ for, $h=1, 2, \cdots, g-1$, where g denotes an integer >1, α being the largest number with that property, then α is called the lower density of A below g. The purpose of the lecture is to discuss the $\alpha+\beta$ theorem which can be stated as follows: If two sequences A and B, formed by integers, both contain 0, and if α and β are their respective lower densities below g, then the lower density of the sum sequence $A+B$ (formed by the numbers which can be written in at least one way as $a+b$, where a belongs to A, b to B) is $\geqq \alpha+\beta$, unless $\alpha+\beta \geqq 1$, when it is 1 and therefore each positive integer $<g$ can be written in at least one way as $a+b$.

Mann has proved this theorem in a more general form: Let A and B be two sequences formed by integers $\geqq 0$, both containing 0, such that

$$(\text{number of elements} \leqq h \text{ of } A)+(\text{number of elements} \leqq h \text{ of } B) \geqq \gamma h$$

for $h=1, 2, \cdots, g-1$, where $\gamma \leqq 1$. Then the lower density of $A+B$ below g is $\geqq \gamma$. Denoting by $A(h)$ the number of elements $<h$ of A, Mann's theorem can be formulated also as follows: If the sequences A and B, formed by integers $\geqq 0$, both contain 0, and if $A(h)>B(h) \geqq \phi(h)+1$, then $(A+B)(h) \geqq \phi(h)$ for $0<h \leqq g$, where $\phi(h)=\gamma(h-1)+1$, $\gamma \leqq 1$. A simple proof of this theorem can be given, showing a more general result, namely that the elements of A and B need not be integers, need not contain 0, and $\phi(h)$ can be any monotonic nondecreasing function satisfying $\phi(h+h') \leqq \phi(h)+\phi(h')$ for $h>0$, $h'>0$, $h+h'<g$.

4. *General mathematics in the college*, by Professor Arthur J. Hall, San Francisco State College, introduced by the Secretary.

The speaker presented an overview of the two-unit general mathematics course now required at San Francisco State College of nearly all candidates for the bachelor's degree. Now in the second experimental year, the course resulted from the re-study of the lower division requirements of the college. The course is related to other phases of the required lower division program. Its purpose is to assist the student to utilize mathematics in personal affairs and in solving social problems, to maintain or improve computational skills, to understand mathematics as a way of thinking, communicating, and solving quantitative problems. The content includes elementary statistics,

budgeting, installment buying, and probability. The text used is *Mathematics for the Consumer* with supplementary materials. The activities provided center around two projects. These projects involve campus problems such as "How do State Students Provide for Their Lunch?", "The Student Body Card Situation and Reasons for Not Buying," and installment buying problems such as "Furnishing a Home" or "Buying a Home." Weekly assignments of exercises from the text are made as the student needs practice on concepts to be used in the project. Students work in groups of 3 or 4 on joint projects. Evaluation reveals that the majority of students feel the projects are well worth their time, the text inappropriate, that the course should be required, and that both students and faculty feel course materials are useful. Standardized tests show statistically significant gains (at 1% level) in computational and reasoning ability in arithmetic.

5. *Calculation of moments for a Cantor-Vitali function*, by Professor G. C. Evans, University of California.

The speaker evaluated the moments $M^n=\int_0^1 x^n f(x)dx$ for the Cantor-Vitali continuous function $f(x)$, with $f(0)=0$, $f(1)=1$, which is obtained by displacing the open middle third of the interval $(0, 1)$ to the level $y=1/2$, the open middle thirds of the remaining intervals to the levels $y=1/4$ and $y=3/4$, respectively, etc., and by making the natural definitions of $y=f(x)$ on the remaining set of content zero. He started with the obvious value $M^0=1/2$ and found $M^1=5/16$, $M^2=11/48$, $M^3=233/1280$, by deriving the recurrence formula $2M^n=1/(n+1)+[(2+M)^n-M^n]/(3^{n+1}-1)$, in which $(2+M)^n$ stands for its binomial expansion beginning 2^nM^0. The formula is obtained by elementary means in terms of the relations $f(3x)=2f(x)$, $f(x+2/3)=1/2+f(x)$, valid for $0\leqq x\leqq 1/3$.

6. *Representation of the integers by positive integers*, by Professor Roy Dubisch, Fresno State College.

There is a 1–1 correspondence between the integers and the positive integers given by $n\leftrightarrow 2n+1$, $(n=0, 1, 2, \cdots)$, and $-n\leftrightarrow 2n$, $(n=1, 2, 3, \cdots)$. In this paper it is observed that this correspondence can be extended to an isomorphism by defining addition and multiplication of positive integers suitably.

E. B. ROESSLER, *Secretary*

APRIL MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met at Central College in Fayette on Friday, April 6, 1951. Professor F. F. Helton, Chairman of the Section, and Professor L. O. Jones, Vice-Chairman of the Section, presided at the morning session. Professor Helton presided at the afternoon session and Professor Margaret F. Willerding, Secretary-Treasurer of the Section, introduced the guest speaker.

Fifty-three persons were in attendance including the following thirty-three members of the association: L. W. Akers, J. J. Andrews, S. Louise Beasley, J. W. Blattner, L. M. Blumenthal, C. A. Bridger, P. B. Burcham, Mary Cummings, C. E. Denny, W. C. Doyle, R. E. Ekstrom, M. B. Evans, G. M. Ewing, C. V. Fronabarger, J. W. Gaddum, M. A. Golub, Nola A. Haynes, F. F. Helton, C. A. Johnson, L. O. Jones, R. E. Lee, Jessie McLean, Ella Marth, R. J. Michel, Marie A. Moore, Margaret Owchar, Gordon Pall, R. M. Rankin, Francis Regan, W. L. Stamey, W. R. Utz, Margaret F. Willerding, J. L. Zemmer.

At the business meeting, the following officers were elected for the coming year: Chairman, Francis Regan, St. Louis University; Vice-Chairman, P. B. Burcham, University of Missouri; Secretary-Treasurer, S. Louise Beasley, Lin-

denwood College.

The annual meeting of 1952 will be held at Lindenwood College, the date to be announced later.

Professor Gordon Pall of Illinois Institute of Technology was the guest speaker. Professor Pall presented a paper on *Generalized Quaternions* at the afternoon session.

The following papers were presented:

1. *The distance set for the Cantor discontinuum*, by Professor W. R. Utz, University of Missouri.

This paper will appear in this MONTHLY.

2. *Modern mathematics for college freshmen*, by Rev. W. C. Doyle, Rockhurst College.

Little of the mathematics developed in the last hundred years appears in standard lower division courses. The paper suggested topics and viewpoints which could be included in college algebra without much revision of content. In analytic geometry two and three dimensions can be treated as a unit by adopting direction numbers (vectors) in place of slope. These number sets easily generalize into matrices. Rotation of axes then becomes a simple application of theory on diameters and the reduction of quadratic forms by means of matrices.

3. *What's wrong with mathematics textbooks?*, by Professor C. A. Johnson, School of Mines and Metallurgy of the University of Missouri.

The quality of pedagogical practices in the teaching of mathematics is still being called into question by prominent educators. It is the function of groups such as the Mathematical Association of America to look into the validity of these criticisms as they apply in every phase of the educational process. The mathematics textbook in itself constitutes one vital phase of this process that appears to need closer examination. The writing of modern mathematics textbooks seems to be vulnerable to attack on the following bases: (1) Failure to follow out the scientific method in the presentation of new theory; (2) Failure to develop a psychologically sound approach to logical organization of the material; (3) Substitution of long lists of "exercises" for real problems; (4) Development of the subject matter in a generally expository and non-reflective manner.

To improve the writing of mathematics textbooks it is recommended that in the future they be the joint effort of the mathematician and the educational psychologist.

4. *An attempt to broaden the background of prospective teachers of mathematics*, by Professor Nola A. Haynes, University of Missouri.

The speaker discussed the subject matter and objectives of a course recently organized at the University of Missouri and designed for prospective teachers of mathematics in secondary schools. The purpose of the course is to provide a broader and more mature understanding of the fundamental concepts of elementary mathematics.

5. *On the maximum likelihood estimation of a single parameter*, by C. A. Bridger, Bureau of Vital Statistics.

Since the estimators obtained by the method of maximum likelihood usually meet Fischer criteria of consistency, efficiency, and sufficiency, they are widely used. The principle of likelihood arose in an attempt to overcome practical difficulties associated with inverse probability. This principle asserts that if A refers to a universe and C to a sample, the probability that C in fact came from A is proportional to the likelihood that C is representative of A. The maximizing of

the likelihood relative to the parameters characterizing A is done in most cases by standard mathematical procedures.

6. *Generalized quaternions*, by Professor Gordon Pall, Illinois Institute of Technology.

With every ternary quadratic form is associated a quaternion algebra, which is closely related to the automorphs of the form. A change of basis of the algebra is accompanied by a corresponding linear transformation of the form. A ring of integral elements with unity is associated in a simple way with a ternary form having integral coefficients. Classes of ideals within such rings are associated with classes of quaternary quadratic forms within the genus of the norm form. The quadratic congruence $\bar{x}ax \equiv b \pmod{p^r}$, where a and b are given pure quaternions, p is a rational prime, and x is a variable quaternion, has interesting properties. For example, if a and b are primitive and of norm zero mod p, and $p+d$, $r=1$, and p is odd, then the congruence is solvable if and only if the trace $2\sum A_{\alpha\beta}a_\alpha b_\beta$ is a quadratic residue mod p, with a special interpretation if the trace is zero mod p. Thus there are two residue systems of such pure vectors mod p. In a similar situation for the prime 2, there are four residue systems. The results for the prime 2 and odd discriminant are given in detail. There is a reciprocity law similar to the Hilbert form of the quadratic reciprocity law. Here primes p behave differently according as the Hansel-Hasse invariant c_p of the ternary form is $+1$ or -1.

MARGARET F. WILLERDING, *Secretary*

MATHEMATICS CONTEST OF THE METROPOLITAN NEW YORK SECTION

The Committee on Contests and Awards of the Metropolitan New York Section of the Association has announced the results of its mathematics contest which was held on May 10, 1951. This contest was open to high school students in the area of the Section. More than 5,000 students participated. It seems evident that the contest is accomplishing its purpose of creating the greatest possible interest in mathematics.

The Association's Bronze Cup was awarded to Brooklyn Technical High School for a score of 402 points out of a possible 450 points. This trophy is given to the school that has the highest total score on the three best papers which it submits.

Certificates of Merit were awarded to the highest ranking student in each county. The following students received these awards: J. S. Lew, Mamaroneck Senior High School, Westchester County; P. H. Monsky, Brooklyn Technical High School, Kings County; Lewis Iscol, Bronx High School of Science, Bronx County; Alicia E. Larde, Marymount School, New York County; T. E. Douglass, Saint Bonaventure's High School, Passaic County; E. J. Stolz, St. Joseph's Normal Institute, Dutchess County; R. A. Howard, Lynbrook High School, Nassau County, G. L. Smith, Chatham High School, Morris County; J. A. Wolf, College High School, Essex County; R. J. Sibner, Forest Hills High School, Queens County; and J. R. Hastings, Rutherford High School, Bergen County.

In addition to the above awards, Honor Keys were awarded to the following seven students who ranked first in their respective schools for the contest of 1950 and also that of 1951: J. S. Lew, Mamaroneck Senior High School, Mamaroneck, New York; P. H. Monsky, Brooklyn Technical High School, Brooklyn,

New York; Stephen Weingram, DeWitt Clinton High School, Bronx, New York; G. L. Smith, Chatham High School, Chatham, New Jersey; Lois B. Mitchell, A. B. Davis High School, Mount Vernon, New York; P. G. Federbush, Weequahic High School, Newark, New Jersey; and Anna Chernovitz, Washington Irving High School, New York, New York. Also individual awards, in the form of a button or pin bearing the facsimile of the seal of the Mathematical Association of America, were given to each student who won first place in his school.

Copies of the contest questions and other information about the contest may be obtained from Professor W. H. Fagerstrom, the City College of the City of New York, New York 31, New York. Professor Fagerstrom has served as Chairman of the Committee on Contests and Awards and the other members of the committee are as follows: Brother Bernard Alfred, Manhattan College; Professor H. F. Fehr, Teachers College, Columbia University; Mr. Julius Freilich, Brooklyn Technical High School; Professor J. J. Kinsella, School of Education, New York University; Professor V. S. Mallory, State Teachers College, Montclair, New Jersey; Mr. K. B. Morgan, Mt. Kisco High School, Mt. Kisco, New York; Mr. Aaron Shapiro, Midwood High School, Brooklyn, New York.

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College Waynesburg, Pennsylvania, May 10, 1952.

ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.

INDIANA, Indiana University, Bloomington, Spring, 1952.

IOWA, Coe College, Cedar Rapids, April 18–19, 1952.

KANSAS

KENTUCKY, University of Kentucky, Lexington.

LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, National Bureau of Standards, Washington, D. C., December 8, 1951.

METROPOLITAN NEW YORK, Spring, 1952.

MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.

MINNESOTA

MISSOURI, Lindenwood College, St. Charles, Spring, 1952.

NEBRASKA

NORTHERN CALIFORNIA, University of California, Berkeley, January 26, 1952.

OHIO, April 19, 1952.

OKLAHOMA, Oklahoma City University, October 12, 1951.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.

PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.

ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May, 1952.

SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.

SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.

TEXAS, East Texas State Teachers College, Commerce, April, 1952.

UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May, 1952.

WISCONSIN

Prentice-Hall
COLLEGE TEXTS

Analytic Geometry and Calculus

By *Lyman M. Kells, U. S. Naval Academy*

Combining the subjects of Analytic Geometry and Calculus, this new text provides the student with a better grasp of both subjects. A tremendous saving of time, by the early introduction of fundamental analytic principles, enables him to go on more quickly to advanced topics and spend less time in review.

A large number of carefully graded problems provide drill work and numerous important applications. Answers and solutions to half the problems are at the back of the book. Remaining answers and solutions are available to the instructor upon adoption of the text. Appendices and index are included.

Published 1950, 623 pages, 6″ x 9″

Analytic Geometry

By *Lyman M. Kells, and Herman C. Stotz, U. S. Naval Academy*

This text has been created to provide a workable background for a thorough understanding of the principles of analytic geometry. Reasoning rather than mere memorization is highlighted throughout. The student is encouraged to recognize a proof as a familiar working principle, rather than just another sound that must be re-learned each time it appears. Ample opportunity is provided for a deep understanding of fundamentals through their application in the solving of ordinary problems. Illustrations are large and well-executed. (Over 1350 carefully graded problems are included)

Published 1949, 280 pages, 6″ x 9″

Send for your copies today!

PRENTICE-HALL INC. PH 70 FIFTH AVE·NY 11·NY

Advanced mathematics texts now published in paperbound editions

Unique new publishing program saves you about 50 per cent

Important monographs on Dover science lists available in limited quantities at savings of $1.00 to $2.70 per title. Ideal for graduate students and teachers.

1. ANALYSIS AND DESIGN OF EXPERIMENTS by H. B. Mann. Variance and variance design analysis. Likelihood ratio tests and tests of linear hypotheses, Latin squares and incomplete balanced block designs, non-orthogonal data, interblock estimates and interblock variance, Galois fields, etc. "Offers mature mathematician with no background in statistics a method of grasping the analysis of variance and variance design within a reasonably short time."—*Engineering Journal.* Index. 5 x 7⅜. vi + 195pp. 3 tables. Cloth ed. $2.95. Special paperbound ed. only $1.25

2. ELEMENTS OF MATHEMATICAL LOGIC by Paul Rosenbloom. Excellent introduction to mathematical logic. Provides mature mathematician with sound knowledge of most important and up-to-date approaches. 5 x 7⅜. 224pp. Exercises. Appendices. Index. Bibliographical remarks. Cloth ed. $2.95. Paperbound $1.25

3. MATHEMATICAL FOUNDATIONS OF STATISTICAL MECHANICS by A. I. Khinchin. Transl. by G. Gamow. "Excellent introduction . . . should prove useful to teachers of statistical mechanics and to mathematicians who would like to get acquainted with problems and methods of this fascinating branch of physics."—*Quarterly of Applied Mathematics.* Appendix. Notations. Index. 5 x 7⅜. viii + 179pp. Cloth ed. $2.95. Paperbound $1.25

4. INTRODUCTION TO THE THEORY OF FOURIER'S SERIES AND INTEGRALS by H. S. Carslaw. Third revised ed. "Needs little introduction . . . much new material has been added . . . clearly and attractively written."—*Nature.* 2 appendices. Index. 5⅜ x 8. xiii + 368pp. 39 ill. Cloth ed. $4.50. Paperbound $1.90

5. CONCISE HISTORY OF MATHEMATICS by Dirk J. Struik. Emphasizes ideas and continuity of mathematics, rather than anecdotal aspects, from Oriental beginnings through 19th century. "Rich in content . . . thoughtful in interpretation."—*U. S. Quarterly Book List.* Two volumes bound in one. Bibliography. Index. 5 x 7⅜. Vol. I: xviii + 123pp. Vol. II: vi + 175pp. 47 ill. Rev. ed. Cloth ed. $3.00. Paperbound $1.60

6. BESSEL FUNCTIONS by Enzo Cambi. Eleven and fifteen-place tables of Bessel functions of first kind to all significant orders. Bibliography. 8½ x 10¾. vi + 160pp. 2 graphs. Cloth ed. $3.95. Paperbound $1.50

7. THEORY OF VIBRATIONS by N. W. McLachlan. Based on an unusually successful series of lectures given at Brown University, this work differs from ordinary texts in 5 different respects: (1) Use of operational calculus for impulses, and its iterated use in forced vibration of strings and bars. (2) Use of Bessel functions to obtain exact solutions for circular plates, bars of variable section; tables of roots included. (3) Inclusion of modern version of certain non-linear problems. (4) Use of frequency spectra. (5) Consideration of electromechanical system. Invaluable to teachers of applied mathematics and physics. 5 x 7⅜. vi + 154pp. Index. Cloth ed. $2.45. Paperbound $1.25

DOVER CLOTHBOUND EDITIONS

8. HIGHER MATHEMATICS FOR STUDENTS OF CHEMISTRY AND PHYSICS by J. W. Mellor. Fourth revised ed. "An eminently readable and thoroughly practical treatise."—*Nature.* 2 appendices. Index. 5½ x 8½. xxix + 641pp. 189 figures. 18 tables. Originally published at $7.00. Clothbound $4.50

9. JACOBIAN ELLIPTIC FUNCTION TABLES by L. M. Milne-Thompson. A guide to practical computation with elliptic functions and integrals together with tables of sn u, cn u, dn u, $Z(u)$. 5 x 7⅜. xi +123pp. Clothbound $2.45

10. LAWS OF THOUGHT by George Boole. "Pure mathematics was discovered by Boole in a work he called The Laws of Thought."—Bertrand Russell. "Boole was inspired if ever a mortal was when he wrote The Laws of Thought."—Dr. E. T. Bell, former president of the Mathematical Association of America. "A work of astonishing originality and power."—*Dictionary of National Biography.* One of the most stimulating mathematical classics ever written, containing numerous leads to future mathematical investigations. Particularly useful in modern algebra, group theory and axiomatics. 5⅜ x 8. 448pp. Clothbound $4.50

11. LINEAR INTEGRAL EQUATIONS by William Vernon Lovitt. Used as the main text at virtually every university or college where this subject is taught. Readable and systematic study of general theory of linear integral equations, with applications in differential equations, calculus of variations and some problems in mathematical physics. 65 special exercises for the student—plus hundreds of demonstration exercises worked out by the author. 5⅜ x 8. xii + 265pp. Index. Preface. Clothbound $3.50

12. NATURE OF PHYSICAL THEORY by P. W. Bridgman. "It can easily be read in about three hours, but it will then demand to be re-read, parts of it several times over."—*Rev. of Scientific Instruments.* Index. 5⅜ x 8. xi + 138pp. Clothbound $2.25

13. NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS by H. Levy and E. A. Baggott. Chapters on graphical integration of differential equations, numerical solution over a limited range, forward integration of first order equations, special methods applicable to linear differential equations of the second order, etc. "Should be in the hands of all those who are in research in applied mathematics and of all those who are interested in the improvement of mathematical teaching."—*Nature.* 5⅜ x 8. viii + 238pp. Clothbound $3.50

14. ORDINARY DIFFERENTIAL EQUATIONS by E. L. Ince. Fourth revised ed. "Notable addition to the mathematical literature in English."—*Bulletin of American Mathematical Society.* 4 appendices. Index. 5½ x 9. viii + 558pp. 18 ill. Originally pub. at $12. Clothbound $4.95

15. SPACE—TIME—MATTER by Hermann Weyl. Transl. by Henry L. Brose from fourth German ed. Analysis of relativity theory, including detailed exposition of tensor calculus. 5⅜ x 8. xviii + 330pp. 15 diagrams. Clothbound $3.95

16. SPHERICAL HARMONICS by T. M. MacRobert. Elementary treatise on harmonic functions with applications. Second rev. ed. "Scholarly treatment of the type of problems arising in a great many branches of theoretical physics and the tools whereby such problems may be attacked."—*Bulletin of American Math. Society.* Index. 5½ x 8½. vi + 372pp. Clothbound $4.50

17. SYMBOLIC LOGIC by Lewis and Langford. Unabridged republication of a great classic. In demand for its historical perspective and original contribution to the subject. 5⅜ x 8. viii + 504pp. Clothbound $4.50

18. THEORY OF GROUPS AND QUANTUM MECHANICS by Hermann Weyl. Transl. from second revised German ed. by H. P. Robertson. Unitary geometry, the Jordan-Holder theorem and its analogues, the Pauli exclusion principle and the structure of the periodic table, spin and valence, etc. "Does not need recommendation."—*Nature.* 2 appendices. Bibliography. Index. 5⅜ x 8. xxii + 422pp. Clothbound $4.50

19. TREATISE ON THE ANALYTICAL DYNAMICS OF PARTICLES AND RIGID BODIES by E. T. Whittaker. Fourth revised ed. "Exhibits great mathematical power and attainments. . . ."—*American Mathematical Monthly.* Index. 6 x 9. xiv + 456pp. Originally published at $6.00. Clothbound $4.50

20. THE FOURIER INTEGRAL AND CERTAIN OF ITS APPLICATIONS by Norbert Wiener. Only book-length study of the Fourier integral as a link between harmonic analysis and mathematical theory, physics and engineering. Bibliography. 5⅜ x 8. xii + 201pp. Clothbound $3.95

EDITIONS LIMITED—DETACH COUPON AND MAIL TODAY

Dover Publications, Dept. 46, 1780 Broadway, N.Y. 19, N.Y.

Send postpaid PAPERBOUND EDITIONS OF BOOKS CIRCLED: 1. 2. 3. 4. 5. 6. 7.

Send postpaid CLOTHBOUND EDITIONS OF BOOKS CIRCLED: 1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14. 15. 16. 17. 18. 19. 20. Unconditional ten-day cash-back guarantee on all editions.

NAME .....

ADDRESS .....

.....

I am enclosing $———— in full payment.

THE APPLETON-CENTURY MATHEMATICS SERIES

The Mathematics of Finance

By FRANKLIN C. SMITH Ph.D.

College of St. Thomas, St. Paul, Minnesota

ASSOCIATE OF THE SOCIETY OF ACTUARIES

Designed specifically as a basic text for college and business school courses in the mathematics of finance, mathematics of investment, financial mathematics, and the like, this book covers systematically the essential topics usually taught in introductory courses. The emphasis throughout the book is on basic principles rather than on special formulas. Teaching aids include 27 diagrams, an abundance and variety of practical exercises, necessary tables, and an index. *356 pp.*, $4.00

APPLETON-CENTURY-CROFTS, INC. 35 West 32nd Street New York 1, New York

The Rhind Mathematical Papyrus

The RHIND MATHEMATICAL PAPYRUS was published under the auspices of the Mathematical Association of America through a gift from the late Arnold Buffum Chace, Chancellor of Brown University. This exposition of one of the very oldest mathematical documents in the world is of value to all students of mathematics and of Egyptian civilization of 4000 years ago. Volume I, 11¼ by 8 inches, 8 + 210 pages, contains the Free Translation, Commentary, and Bibliography of Egyptian Mathematics; Volume II, 11¼ by 14¼ inches, contains 140 photographic plates in original colors, black and red, with Text and Introductions, and Literal Translation. The price to members of the Association is $20 for the set; to non-members $25 for the set. Members may purchase sets through the office of the Secretary of the **Mathematical Association of America, University of Buffalo, Buffalo 14, New York.** Non-members must purchase copies from the **Open Court Publishing Company, La Salle, Illinois.**

New Addison-Wesley Mathematics Books

CALCULUS AND ANALYTIC GEOMETRY

By GEORGE B. THOMAS, JR., *Massachusetts Institute of Technology*

Motivation is the keynote of this new textbook developed at the Massachusetts Institute of Technology in actual classroom use during the past two years. Designed primarily for scientists and engineers the text stresses many applications to geometry and mechanics with integration introduced at an early stage.

Clothbound, Illustrated, 756 pages.

THE TEACHING OF MATHEMATICS

By DAVID R. DAVIS, *State Teachers College, Montclair, New Jersey*

A new text for methods courses in mathematics. Primarily for the preparation of teachers of secondary school mathematics, it offers a thorough treatment of teaching techniques correlated and integrated with a discussion of the subject matter of such courses.

Clothbound, Illustrated, 384 pages.

THE PREPARATION OF PROGRAMS FOR AN ELECTRONIC DIGITAL COMPUTER

By M. V. WILKES, D. J. WHEELER, and STANLEY GILL, *Cambridge University, England*

For those now using electronic digital computers, for those putting new machines into operation, and for those wishing to assess the possible application of such computers to their own problems, this book discusses new methods of preparing problems for computation, with special reference to EDSAC.

Clothbound, Illustrated, 184 pages.

THE CLASSICAL THEORY OF FIELDS

By L. LANDAU *and* E. LIFSHITZ, *Academy of Sciences of the U.S.S.R. Translated by* MORTON HAMERMESH, *Argonne National Laboratory*

A systematic presentation of the theory of electromagnetic and gravitational fields for courses on the graduate level. The complete, logically connected theory of the electromagnetic field includes the special theory of relativity which has been used as the basis of this presentation.

Clothbound, Illustrated, 432 pages.

THE ALGEBRA OF VECTORS AND MATRICES

By T. L. WADE, *Florida State University*

A textbook on the senior-graduate level which presents the elementary expositions of matrix and vector algebra articulated with the group, integral domain, field, ring, basis, dimension, and isomorphic concepts of modern algebraic theory.

Clothbound, Illustrated, c. 200 pages.

CLASSICAL MECHANICS

By HERBERT GOLDSTEIN, *Harvard University*

A text for mechanics courses on the senior-graduate level. The presentation is designed to furnish the student with classical mechanics in the formulations most appropriate to the study of modern physics, especially quantum mechanics.

Clothbound, Illustrated, 399 pages

Examination copies cheerfully sent upon request.

Addison-Wesley Press, Inc. CAMBRIDGE 42 MASSACHUSETTS

Princeton Books in the Annals of Mathematics Studies

EXISTENCE THEOREMS IN PARTIAL DIFFERENTIAL EQUATIONS

By Dorothy L. Bernstein

Contents: I. Introduction. II. The problem of Cauchy for first order differential equations. III. Second order differential equations of elliptic, hyperbolic, and parabolic types; linear, quasi-linear, and general order equations. IV. Differential equations of order $n > 2$.

Study No. 23 — *225 pages* — *$2.50*

CONTRIBUTIONS TO THE THEORY OF GAMES

Edited By H. W. Kuhn & A. W. Tucker

A collection of recent research in the theory of games of strategy, continuing the theory as developed by von Neumann and Morgenstern in their *Theory of Games and Economic Behavior.*

Study No. 24 — *217 pages* — *$3.00*

CONTRIBUTIONS TO FOURIER ANALYSIS

By A. Zygmund, W. Transue, M. Morse, A. P. Calderon, & S. Bochner

Contents: I. Localization of best approximation. II. Dirichlet Problem for domains bounded by spheres. III. The Frechet variation and Pringsheim convergence of double Fourier systems. IV. Boundary values of functions of several complex variables. V. Trigonometric interpolation.

Study No. 25 — *192 pages* — *$3.00*

A THEORY OF CROSS-SPACES

By Robert Schatten

Contents: I. The Algebra of Expressions $\Sigma_{i=1}^{n} f_i \otimes g_i$. II. Crossnorms. III. Cross-spaces of Operators. IV. Ideal of Operators. V. Crossed Unitary Spaces. Appendices.

Study No. 26 — *161 pages* — *$2.50*

ISOPERIMETRIC INEQUALITIES IN MATHEMATICAL PHYSICS

By G. Pólya & G. Szegö

Analogous statements in mathematical physics of the isoperimetric theorem are collected and discussed. Original contributions are included.

Study No. 27 — *279 pages* — *$3.00*

PRINCETON UNIVERSITY PRESS, Princeton, N.J.

Outstanding McGRAW-HILL Books

PLANE AND SPHERICAL TRIGONOMETRY. New 3rd edition

By LYMAN M. KELLS, WILLIS F. KERN, and JAMES R. BLAND, United States Naval Academy. 408 pages (with tables), $3.75

While all the good features of the past edition have been retained, improvements were effected in the new edition by eliminating items which did not have a definite purpose, changing the logical as well as the pedagogical order, by simplification of major developments, and by the addition of new theory, new proofs, and new problems.

PLANE TRIGONOMETRY. New 3rd Edition

By LYMAN M. KELLS, WILLIS F. KERN, and JAMES R. BLAND. 319 pages (with tables), $3.50

This text, published as a separate volume, consists of the first section of *Plane and Spherical Trigonometry* covering only plane trigonometry. Elementary applications of fundamental ideas in everyday situations illustrate both principles and application.

ADVANCED ENGINEERING MATHEMATICS

By C. R. WYLIE, University of Utah. 640 pages, $7.50

Provides an introduction to those fields of advanced mathematics which are currently of engineering significance. Covers such topics as ordinary and partial differential equations, Fourier series and the Fourier integral, vector analysis, numerical solution of equations and systems of equations, finite differences, least squares, etc.

THEORY OF ELASTICITY. New 2nd edition

By S. TIMOSHENKO, Stanford University, and J. N. GOODIER, Stanford University. *Engineering Societies Monographs.* 500 pages, $9.50

A modernization of this very successful text, the second edition, contains new and revised chapters on the photoelastic method, complex variables, and thermal stress. Shorter additions include discussions of: the theory of the strain gage rosette, gravity stresses, Saint-Venant's Principle, the components of rotation, the Reciprocal theorem, etc.

GRAPHIC AIDS IN ENGINEERING COMPUTATION

By R. P. HOELSCHER, S. H. PIERCE, University of Illinois, and J. NORMAN ARNOLD, Purdue University. Ready in December.

A text designed for courses covering the principles and methods of the slide rule and the construction of alignment charts. Includes chapters on the standard slide rules, the derivation of empirical equations from laboratory or field data, the construction of nomographs by geometric methods and with determinants, the construction of special slide rules, the solutions of problems by graphical differentiation and integration, and advanced work on movable scale charts.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc.

330 WEST 42nd STREET • NEW YORK 36, N. Y.

val

ere
se,
op-

ith

nd
of

of
ns,
ons

ity.

re-
ter
es,

due

the
the
no-
ide
ced

1951 HEATH COLLEGE MATHEMATICS TEXTS

WILLIAM L. HART: COLLEGE TRIGONOMETRY

A substantial treatment of plane and spherical trigonometry of moderate length, incorporating distinctly collegiate viewpoints. Emphasizes analytic trigonometry, oriented for application in later mathematics; presents a mature and well-rounded treatment of numerical plane trigonometry; and offers a satisfactory foundation in spherical trigonometry, including a reasonable number of elementary applications. Text pages: *Plane Trigonometry,* 151; *Complex Numbers and Appendix,* 21; *Spherical Trigonometry,* 35; *Logarithmic and Trigonometric Tables,* 130. *$3.50*

TOMLINSON FORT: CALCULUS

The first chapter on infinite series—the best introduction to the notion of limit, basic to the study of calculus; free use of series throughout, particularly power series, making possible a simple and straightforward treatment of Taylor's series; the logarithm introduced as a direct function defined by an integral; designed for regular college courses, providing also a thorough grounding for engineering students; mathematically rigorous but not difficult, with more topics than usually given in the calculus course, facilitating a suitable selection. Text pages: 508; *Tables,* 20. *$4.75*

D. C. HEATH AND COMPANY

COLLEGE DEPARTMENT: 285 Columbus Avenue, Boston 16, Massachusetts

MACMILLAN

for fall classes

PRIMER OF COLLEGE MATHEMATICS

JOHN RANDOLPH

Professors Davis, Demers, and Beesley of the Department of Mathematics and Mechanics at the University of Nevada report: *"We recommend this text for any unified freshman mathematics course. We find the material presented in a modern spirit. The treatment of mathematical induction is superior to any that we have seen in a freshman text."* *1950* *$4.75*

PLANE AND SPHERICAL TRIGONOMETRY

MOSES RICHARDSON

This text is characterized by unusually lucid and full explanations of concepts, reasoning, and motivation. Correct proofs are given and throughout there is an emphasis on logical thinking and its usefulness in trigonometry.

1950 *With tables—$3.75* *Without tables—$3.40*

ADVANCED DYNAMICS—2 vols.

E. HOWARD SMART

The work offers a valuable course of Dynamics for students of both pure and applied mathematics. It is a very competent account of dynamics developed in such a manner as to be particularly suitable for honours students in universties. The set consists of volume I, Dynamics of a Particle, and volume II, Dynamics of a Solid Body. *1951—$12.00 the set.*

THE MACMILLAN COMPANY

60 FIFTH AVE., NEW YORK 11, N.Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

Ayuda

9

LOGY DEPT.

PUBLIC LIBRARY
NOV 19 1951
DETROIT

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 58 NUMBER 9

CONTENTS

Brook Taylor and the Mathematical Theory of Linear Perspective . . . . . P. S. JONES 597
The Problem of n Liars and Markov Chains . . . . WILLIAM FELLER 606
A Simple Iterative Solution for Certain Simultaneous Quadratic Equations . . . . . W. S. LOUD 609
Mathematical Notes . . . . . A. J. HOFFMAN, ALBERT WILANSKY, ROY DUBISCH, E. M. WRIGHT, LUTHER CHEO, VICTOR THÉBAULT 613
Classroom Notes . . . D. E. RICHMOND, A. ERDÉLYI, A. WOLINSKY 622
Elementary Problems and Solutions 631
Advanced Problems and Solutions 636
Recent Publications 644
Clubs and Allied Activities 648
News and Notices 653
Mathematical Association of America 659
Thirty-second Summer Meeting of the Association 659
March Meeting of the Michigan Section 663
April Meeting of the Ohio Section 668
Calendar of Future Meetings 670

NOVEMBER 1951